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Technical Details of Multiscale Bootstrap for Singular Surfaces

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Technical Details of Multiscale Bootstrap for Singular Surfaces

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Introduction

▪ Summary

This supplementary document gives technical details of the paper "Approximately Unbiased Tests for Singular Surfaces via Multiscale Bootstrap Resampling", which is referred to as the main paper below. All the numerical calculations of the main paper are exactly reproduced in this document, except for the phylogeny example of Section 2.

In the PDF version of this document, some parts are hidden to save the space. The *Mathematica* notebook version includes all the parts. It took an hour to run all the programs.

▪ Sections

1. Introduction

2. Phylogeny Example:
Corresponds to Section 2 of the main paper. Another implementation in R language was used there. A suttle differences in MLEs and p-values are observed due to the numerical optimization.

3. Three Populations Example:
Corresponds to Section 3 of the main paper. Geometry of multiple comparisons methods is shown.

4. Filter Representations:
Corresponds to a part of Section 5 of the main paper. The linear theory for normal model is discussed, and details of proofs are given.

5. Surfaces of Revolution Example:
Corresponds to a part of Section 5 of the main paper. The example of Section 5.3 is discussed.

■ Preliminary

Suppress spelling error messages

```
Off[General::spell]; Off[General::spell1]
```

Suppress numerical integration error messages (for "Three populations example")

```
Off[NIntegrate::inum]
```

```
Off[NIntegrate::nintp]
```

```
Off[NIntegrate::nlim]
```

Loading graphics library (for "Phylogeny example")

```
<< Graphics`Graphics`
```

Set your own working directory below

```
SetDirectory["C:\\Documents and Settings\\shimo\\My Documents\\"]
```

```
C:\Documents and Settings\shimo\My Documents
```

Phylogeny Example

■ Multiscale Bootstrap Procedure

■ Preliminary

Normal distribution , density, and quantile functions.

$$\text{Fnorm}[z\_] := \frac{1}{2}\left(1 + \text{Erf}\left[\frac{z}{\sqrt{2}}\right]\right)$$

$$\text{Fnorm2}[z\_] := \text{If}\left[z < -7,\ e^{-\frac{z^2}{2}}\left(-\frac{3}{\sqrt{2\pi}\, z^5} + \frac{1}{\sqrt{2\pi}\, z^3} - \frac{1}{\sqrt{2\pi}\, z}\right),\ \text{Fnorm}[z]\right]$$

$$\text{dnorm}[z\_] := \frac{1}{\sqrt{2\pi}}\ \text{Exp}\left[-\frac{z^2}{2}\right];$$

$$\text{Qnorm}[y\_] := \sqrt{2}\ \text{InverseErf}[2y - 1]$$

■ Parametric Models

Parameter vector is $\beta = (\beta_0, \beta_1, \ldots, \beta_{k-1}) = \{\beta[[1]], \beta[[2]], \ldots, \beta[[k]]\}$.

We use $s = \sigma^2$ in the below.

ψ_k model definition :

$$\text{psik}[k\_][\beta\_, s\_] := \sum_{j=0}^{k-1} \beta[[j+1]]\, s^j$$

ψ_s model definition :

$$\text{psis}[\beta\_, s\_] := \beta[[1]] + \frac{\beta[[2]]\, s}{1 + \beta[[3]]\sqrt{s}}$$

■ Log-likelihood function

The lik function for psi model is

$$\text{lik}[\text{psi\_}, \beta\_][\text{si\_}, \text{ci\_}, \text{c0\_}] := \sum_{i=1}^{\text{Length}[\text{ci}]} \left(\text{ci}[[i]]\,\text{Log}\left[\text{Fnorm}\left[-\frac{\text{psi}[\beta, \text{si}[[i]]]}{\sqrt{\text{si}[[i]]}}\right]\right] + (\text{c0} - \text{ci}[[i]])\,\text{Log}\left[\text{Fnorm}\left[\frac{\text{psi}[\beta, \text{si}[[i]]]}{\sqrt{\text{si}[[i]]}}\right]\right]\right)$$

The lik value for binomial model without any constraint is

$$\text{lik0}[\text{ci\_}, \text{c0\_}] := \sum_{i=1}^{\text{Length}[\text{ci}]} \left(\text{ci}[[i]]\,\text{Log}\left[\frac{\text{ci}[[i]]}{\text{c0}}\right] + (\text{c0} - \text{ci}[[i]])\,\text{Log}\left[1 - \frac{\text{ci}[[i]]}{\text{c0}}\right]\right)$$

We define rss = 2*(lik0 - lik) as

$$\text{xLogyz}[x\_, y\_, z\_] := \text{If}\left[x == 0, 0, -x\,\text{Log}\left[\frac{z}{y}\right]\right]$$

$$\text{rss}[\text{psi\_}, \beta\_][\text{si\_}, \text{ci\_}, \text{c0\_}] := \sum_{i=1}^{\text{Length}[\text{ci}]} 2\left(\text{xLogyz}\left[\text{ci}[[i]], \frac{\text{ci}[[i]]}{\text{c0}}, \text{Fnorm2}\left[-\frac{\text{psi}[\beta, \text{si}[[i]]]}{\sqrt{\text{si}[[i]]}}\right]\right] + \text{xLogyz}\left[\text{c0} - \text{ci}[[i]], 1 - \frac{\text{ci}[[i]]}{\text{c0}}, \text{Fnorm2}\left[\frac{\text{psi}[\beta, \text{si}[[i]]]}{\sqrt{\text{si}[[i]]}}\right]\right]\right)$$

AIC value (as the difference from the unconstraint binomial model) is calculated from the output of FindMinimum

```
calcaic[opt_, ci_] := opt[[1]] + 2 * (Length[opt[[2]]] - Length[ci])
```

■ Corrected p-value

General formula for the corrected z-value is given by

$$\text{zmb}[k\_, \text{s0\_}][\text{psi\_}, \beta\_] := \text{Module}\left[\{s\}, \text{Sum}\left[D[\text{psi}[\beta, s], \{s, j\}]\,\frac{(-1-s)^j}{j!}, \{j, 0, k-1\}\right] /.\ s \to \text{s0}\right]$$

Explicit forms of the z-values for specified models are given as follows

model ψ_1 : $k = 1$

```
Table[zmb[k, σ0^2][psik[1], {β0}], {k, 1, 1}] // TableForm
```

$\beta 0$

model ψ_2 : k = 1, 2

```
Table[Simplify[zmb[k, σ0^2][psik[2], {β0, β1}]], {k, 1, 2}] // TableForm
```

$\beta 0 + \beta 1\, \sigma_0^2$
$\beta 0 - \beta 1$

model ψ_3 : k = 1, 2, 3

```
Table[FullSimplify[zmb[k, σ0^2][psik[3], {β0, β1, β2}]], {k, 1, 3}] // TableForm
```

$\beta 0 + \beta 1\, \sigma_0^2 + \beta 2\, \sigma_0^4$
$\beta 0 - \beta 1 - \beta 2\, \sigma_0^2\, (2 + \sigma_0^2)$
$\beta 0 - \beta 1 + \beta 2$

model ψ_4 : k = 1, 2, 3, 4

```
Table[Collect[zmb[k, σ0^2][psik[4], {β0, β1, β2, β3}], {β0, β1, β2, β3}, Simplify],
  {k, 1, 4}] // TableForm
```

$\beta 0 + \beta 1\, \sigma_0^2 + \beta 2\, \sigma_0^4 + \beta 3\, \sigma_0^6$
$\beta 0 - \beta 1 - \beta 2\, \sigma_0^2\, (2 + \sigma_0^2) - \beta 3\, \sigma_0^4\, (3 + 2\, \sigma_0^2)$
$\beta 0 - \beta 1 + \beta 2 + \beta 3\, \sigma_0^2\, (3 + 3\, \sigma_0^2 + \sigma_0^4)$
$\beta 0 - \beta 1 + \beta 2 - \beta 3$

model $\psi_{s\,:k}$ = 1, 2, 3, 4

```
Table[FullSimplify[zmb[k, σ0^2][psis, {β0, β1, β2}], σ0 > 0], {k, 1, 4}] // TableForm
```

$\beta 0 + \frac{\beta 1\, \sigma_0^2}{1+\beta 2\, \sigma_0}$

$\frac{2\, (\beta 0-\beta 1)+\beta 2\, \sigma_0\, (4\, \beta 0-\beta 1+2\, \beta 0\, \beta 2\, \sigma_0+\beta 1\, \sigma_0^2)}{2\, (1+\beta 2\, \sigma_0)^2}$

$\frac{-3\, \beta 1\, \beta 2+\sigma_0\, (8\, \beta 0-\beta 1\, (8+\beta 2^2)+\beta 2\, \sigma_0\, (6\, (4\, \beta 0-3\, \beta 1)+\sigma_0\, (6\, (4\, \beta 0-\beta 1)\, \beta 2+\sigma_0\, (\beta 1+8\, \beta 0\, \beta 2^2+3\, \beta 1\, \beta 2\, \sigma_0))))}{8\, \sigma_0\, (1+\beta 2\, \sigma_0)^3}$

$\frac{-\beta 1\, \beta 2+\sigma_0\, (-4\, \beta 1\, \beta 2^2+\sigma_0\, (-\beta 1\, \beta 2\, (9+\beta 2^2)+\sigma_0\, (4\, (4\, \beta 0-\beta 1\, (4+5\, \beta 2^2))+\beta 2\, \sigma_0\, (64\, \beta 0-5\, \beta 1\, (11+\beta 2^2)+\sigma_0\, (12\, (8\, \beta 0-5\, \beta 1)\, \beta 2+\sigma_0\, (\beta 1+64\, \beta 0\, \beta 2^2-}{16\, \sigma_0^3\, (1+\beta 2\, \sigma_0)^4}$

The above expressions for ψ_s can be rearranged as follows.

```
Table[β0 - (1 + σ0 β2)^-k
   Collect[FullSimplify[-(1 + σ0 β2)^k (zmb[k, σ0^2][psis, {β0, β1, β2}] - β0), σ0 > 0],
    {β1, β2}, FullSimplify[#, σ0 > 0] &], {k, 1, 4}] // TableForm
```

$\beta 0 + \frac{\beta 1\, \sigma_0^2}{1+\beta 2\, \sigma_0}$

$\beta 0 - \frac{\beta 1\, \left(1+\frac{1}{2}\, \beta 2\, (\sigma_0-\sigma_0^3)\right)}{(1+\beta 2\, \sigma_0)^2}$

$\beta 0 - \frac{\beta 1\, \left(1+\frac{1}{8}\, \beta 2^2\, (1+6\, \sigma_0^2-3\, \sigma_0^4)+\frac{\beta 2\, (3+18\, \sigma_0^2-\sigma_0^4)}{8\, \sigma_0}\right)}{(1+\beta 2\, \sigma_0)^3}$

$\beta 0 - \frac{\beta 1\, \left(1+\frac{1}{4}\, \beta 2^2\, \left(5+\frac{1}{\sigma_0^2}+15\, \sigma_0^2-\sigma_0^4\right)+\frac{\beta 2^3\, (1+5\, \sigma_0^2+15\, \sigma_0^4-5\, \sigma_0^6)}{16\, \sigma_0}+\frac{\beta 2\, (1+9\, \sigma_0^2+55\, \sigma_0^4-\sigma_0^6)}{16\, \sigma_0^3}\right)}{(1+\beta 2\, \sigma_0)^4}$

■ Narrow σ Range Case

■ parameters

sample size of original data

```
n0 = 3414;
```

sample sizes of bootstrap replicates

```
ni = {13656, 10838, 8602, 6828, 5419, 4301, 3414, 2709, 2150, 1707, 1354, 1075, 853};
```

Thus scales are (in σ^2)

```
si = n0/ni; si // N
```

```
{0.25, 0.315003, 0.396884, 0.5, 0.630006,
 0.793769, 1., 1.26024, 1.58791, 2., 2.52142, 3.17581, 4.00234}
```

number of bootstrap samples

```
c0 = 100000;
```

counts of tree-7

```
ci = {20, 57, 140, 308, 582, 971, 1510, 2068, 2713, 3325, 4149, 4496, 5137};
```

■ Maximum likelihood estimate

■ Corrected p-values and AICs ($\sigma_0 = 1$)

model ψ_1 : k = 1

```
Table[Fnorm[-zmb[k, 1][psik[1], {β0}]], {k, 1}] /. opt1[[2]]
```

```
{0.00634867}
```

```
calcaic[opt1, ci]
```

```
12708.7
```

model ψ_2 : k = 1, 2

```
Table[Fnorm[-zmb[k, 1][psik[2], {β0, β1}]], {k, 1, 2}] /. opt2[[2]]
```

```
{0.0152759, 0.0832069}
```

```
calcaic[opt2, ci]
```

```
72.1499
```

model ψ_3 : k = 1, 2, 3

```
Table[Fnorm[-zmb[k, 1][psik[3], {β0, β1, 0.01 β2}]], {k, 1, 3}] /. opt3[[2]]
```

{0.0150787, 0.101705, 0.123043}

```
calcaic[opt3, ci]
```

−5.88955

model ψ_4 : k = 1, 2, 3, 4

```
Table[Fnorm[-zmb[k, 1][psik[4], {β0, β1, 0.01 β2, 0.01 β3}]], {k, 1, 4}] /. opt4[[2]]
```

{0.0148836, 0.103404, 0.141349, 0.15179}

```
calcaic[opt4, ci]
```

−7.11184

model ψ_s : k = 1, 2, 3, 4

```
Table[Fnorm[-zmb[k, 1][psis, {β0, β1, β2}]], {k, 1, 4}] /. opts[[2]]
```

{0.0148513, 0.101876, 0.142387, 0.171631}

```
calcaic[opts, ci]
```

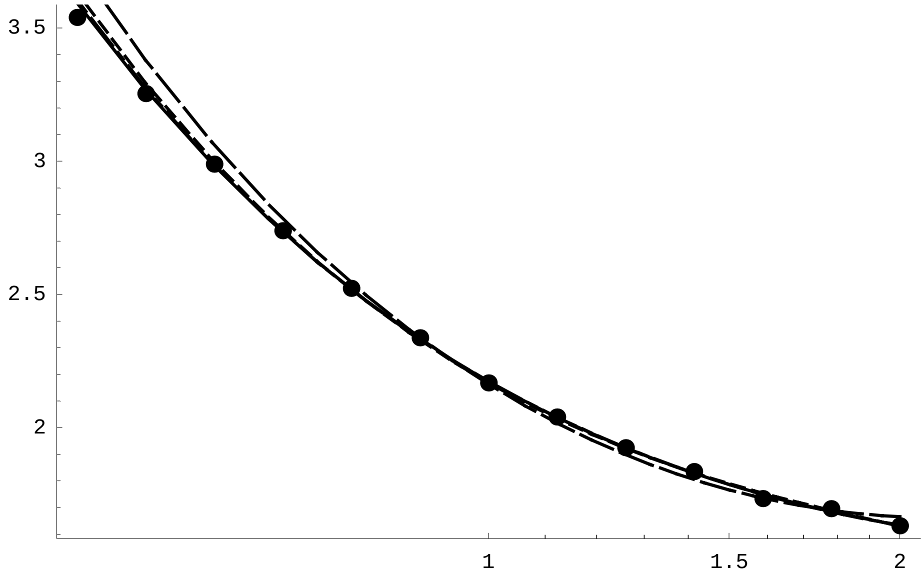
−10.2022

■ plots

■ Summary plot

```
gnarrow = {gpsi0, gpsi2, gpsi3, gpsi4, gpsis};
```

```
Show[gnarrow]
```



- Graphics -

```
Save["tech200605-gnarrow.txt", gnarrow]
```

■ Wide σ Range Case

■ parameters

sample size of original data

```
n0 = 3414;
```

sample sizes of bootstrap replicates

```
ni = {873984, 346840, 137643, 54624, 21677, 8602, 3414, 1354, 537, 213, 84, 33, 13};
```

Thus scales are (in σ^2)

```
si = n0/ni; si // N
```

```
{0.00390625, 0.00984316, 0.0248033, 0.0625, 0.157494,
 0.396884, 1., 2.52142, 6.35754, 16.0282, 40.6429, 103.455, 262.615}
```

number of bootstrap samples

```
c0 = 100000;
```

counts of tree-7

```
ci = {0, 0, 0, 0, 1, 146, 1499, 4017, 5942, 6703, 6794, 5827, 5395};
```

■ Maximum likelihood estimate

■ Corrected p-values and AICs ($\sigma_0 = 1$)

model ψ_1 : k = 1

```
Table[Fnorm[-zmb[k, 1][psik[1], {β0}]], {k, 1}] /. opt1[[2]]
```

```
{0.000010750860571855169671}
```

```
calcaic[opt1, ci]
```

```
175038.83956737111598
```

model ψ_2 : k = 1, 2

```
Table[Fnorm[-zmb[k, 1][psik[2], {β0, β1}]], {k, 1, 2}] /. opt2[[2]]
```

```
{0.002623780419286688, 0.005308899594617150}
```

```
calcaic[opt2, ci]
```

```
16987.975437371656165
```

model ψ_3 : k = 1, 2, 3

```
Table[Fnorm[-zmb[k, 1][psik[3], {β0, β1, 0.01 β2}]], {k, 1, 3}] /. opt3[[2]]
```

{0.00713357, 0.0196665, 0.0197451}

```
calcaic[opt3, ci]
```

3969.3413064305545959

model ψ_4 : k = 1, 2, 3, 4

```
Table[Fnorm[-zmb[k, 1][psik[4], {β0, β1, 0.01 β2, 0.0001 β3}]], {k, 1, 4}] /. opt4[[2]]
```

{0.0109968, 0.0387904, 0.0393685, 0.0393712}

```
calcaic[opt4, ci]
```

1140.1052133323919406

model ψ_S : k = 1, 2, 3, 4

```
Table[Fnorm[-zmb[k, 1][psis, {β0, β1, β2}]], {k, 1, 4}] /. opts[[2]]
```

{0.014789932138728461, 0.096755568589506861,
 0.132491786613547584, 0.157486798706495597}

```
calcaic[opts, ci]
```

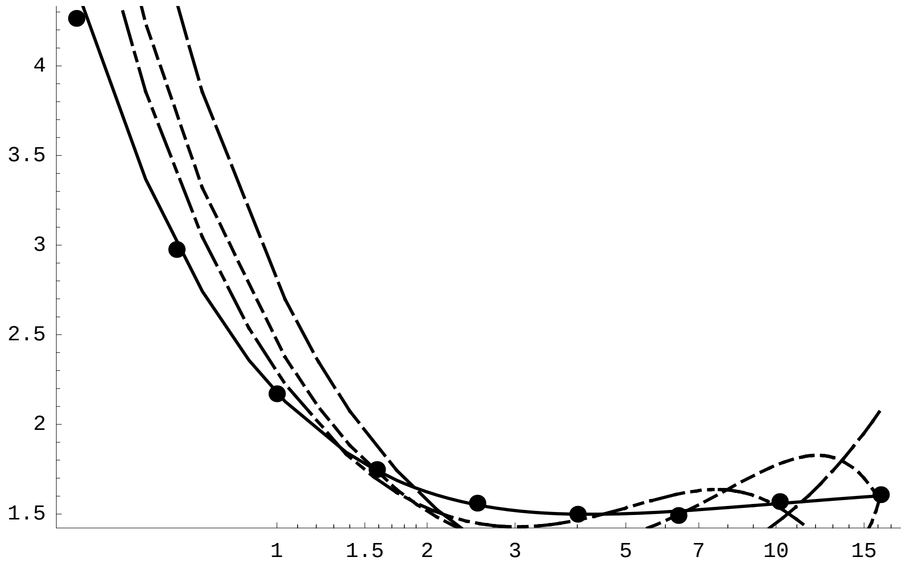
6.025429393395275322

- **plots**

- **Summary plot**

```
gwide = {gpsi0, gpsi2, gpsi3, gpsi4, gpsis};
```

```
Show[gwide]
```



- Graphics -

```
Save["tech200605-gwide.txt", gwide]
```

Three Populations Example

Rejection Regions

Preliminary

$$\text{Fnorm}[z\_] := \frac{1}{2}\left(1+\text{Erf}\left[\frac{z}{\sqrt{2}}\right]\right)$$

Fnorm[z]

$$\frac{1}{2}\left(1+\text{Erf}\left[\frac{z}{\sqrt{2}}\right]\right)$$

Fnorm[-z]

$$\frac{1}{2}\left(1-\text{Erf}\left[\frac{z}{\sqrt{2}}\right]\right)$$

$$\text{dnorm}[z\_] := \frac{1}{\sqrt{2\pi}}\,\text{Exp}\left[-\frac{z^2}{2}\right];$$

$$\text{Qnorm}[y\_] := \sqrt{2}\ \text{InverseErf}[2\,y-1]$$

Bootstrap Probability

Hypothesis region is H={(y1,y2) : y2 ≤ -a Abs[y1]}

(y1, y2) is distributed as $N_2((\mu 1, \mu 2), \sigma^2 I)$

$\text{Fbp} = P_1((y1, y2) \in H) = P_1(y2 \le -a\,\text{Abs}[y1] \mid (\mu 1, \mu 2))$

$$\text{Fbp}[\mu 1\_, \mu 2\_, a\_] := \text{Block}\Big[\{\text{y1fbp}\},\ \text{NIntegrate}\Big[\frac{\left(e^{-\frac{1}{2}(\text{y1fbp}-\mu 1)^2}+e^{-\frac{1}{2}(\text{y1fbp}+\mu 1)^2}\right)\left(1-\text{Erf}\left[\frac{a\,\text{y1fbp}+\mu 2}{\sqrt{2}}\right]\right)}{2\sqrt{2\pi}}, \{\text{y1fbp}, 0, \infty\}\Big]\Big]$$

Zbp=-Qnorm[Fbp]

$$\text{Zbp}[\mu 1\_, \mu 2\_, a\_] := -\sqrt{2}\ \text{InverseErf}[2\,\text{Fbp}[\mu 1, \mu 2, a]-1]$$

For $\sigma^2 = s$; $\text{Zbp} = -\text{Qnorm}[P_\sigma((y1, y2) \in H)]$

$$\text{Zbp}[\mu 1\_, \mu 2\_, a\_, s\_] := \sqrt{s}\ \text{Zbp}\left[\frac{\mu 1}{\sqrt{s}}, \frac{\mu 2}{\sqrt{s}}, a\right]$$

Let Rbp be the rejection surface: Zbp[μ1,-Rbp,a]==z. For μ1 larger than x0, an approximation formula is used.

The rejction region is R={(y1,y2) : y2>-Rbp[y1]}

```
Rbp[μ1_, z_, a_, x0_] := Block[{u2rbp}, If[μ1 < x0,
   u2rbp /. FindRoot[Zbp[μ1, -u2rbp, a] - z, {u2rbp, a μ1 - z Sqrt[1 + a^2]}], a μ1 - z Sqrt[1 + a^2]]]
```

Let Pbp be the rejection probability

Pbp $= P_1((y1, y2) \in R \mid (\mu1, \mu2)) = P_1(y2 > \text{-Rbp[y1,z,a,x0]} \mid (\mu1,\mu2))$

```
Pbp[μ1_, μ2_, z_, a_, x0_] := Block[{x1pbp}, 1 - NIntegrate[1/(2 Sqrt[2 π])
   ((E^(-1/2 (x1pbp-μ1)^2) + E^(-1/2 (x1pbp+μ1)^2)) (1 - Erf[(Rbp[x1pbp, z, a, x0] + μ2)/Sqrt[2]])), {x1pbp, 0, ∞}]]
```

■ Multiscale Bootstrap

corrected z-value is defined by

$$Z_{k,\sigma0}[\mu1, \mu2, a] = \sum_{j=0}^{k-1} \left(\frac{d^j\, Zbp[\mu1, \mu2, a, s]}{d\, s^j} \right)_{\sigma0^2} \frac{(-1 - \sigma0^2)^j}{j\,!}$$

Here $s0 = \sigma0^2$.

```
Zau[μ1_, μ2_, a_, s0_, k_] :=
 Sum[D[Zbp[μ1, μ2, a, s], {s, j}] (-1 - s)^j/j!, {j, 0, k - 1}] /. s → s0
```

Prepare faster versions (This takes about 5 mins)

```
Do[Zau[k][μ1_, μ2_, a_, s0_] = Simplify[Zau[μ1, μ2, a, s0, k]], {k, 1, 4}]

Save["tech200605-zau.txt", Zau]
```

The above result can be loaded later by ;
<<"tech200605-zau.txt"

The suffix "au" is used only internally. For multiscale bootstrap, we use suffix "mb" below.

Corrected z-value

```
Zmb[k_, s0_][μ1_, μ2_, a_] := Zau[k][μ1, μ2, a, s0]
```

Corrected p-value

```
Fmb[k_, s0_][μ1_, μ2_, a_] := Fnorm[-Zau[k][μ1, μ2, a, s0]]
```

Let Rmb be the rejection surface: Zmb[μ1,-Rmb,a]==z. For μ1 larger than x0, an approximation formula is used.

The rejction region is R={(y1,y2) : y2>-Rmb[y1]}

```
Rmb[k_, s0_][x1_, z_, a_, x0_] :=
 Block[{u2rmb}, If[x1 < x0, u2rmb /. FindRoot[Zmb[k, s0][x1, -u2rmb, a] - z,
    {u2rmb, a x1 - z Sqrt[1 + a^2] - 3, a x1 - z Sqrt[1 + a^2] + 3}], a x1 - z Sqrt[1 + a^2]]]
```

Let Pmb be the rejection probability

Pmb $= P_1$((y1, y2) $\in R$ | (μ1, μ2)) $= P_1$(y2 > -Rmb[y1,z,a,x0] | (μ1,μ2))

```
Pmb[k_, s0_][μ1_, μ2_, z_, a_, x0_] :=
 Block[{x1pmb}, 1 - NIntegrate[1/(2 Sqrt[2 π]) (E^(-1/2 (x1pmb-μ1)^2) + E^(-1/2 (x1pmb+μ1)^2))
     (1 - Erf[(Rmb[k, s0][x1pmb, z, a, x0] + μ2)/Sqrt[2]]), {x1pmb, 0, ∞}]]
```

Normal Test

z-value is given by

```
Znt[y1_, y2_, a_ /; a > 0] := (y2 + Abs[a y1] ) / Sqrt[1 + a0^2]

Znt[y1_, y2_, a_ /; a <= 0] := (y2 - Abs[a y1]) / Sqrt[1 + a0^2]
```

Then p-value is

```
Fnt[y1_, y2_, a_] := Fnorm[-Znt[y1, y2, a]]
```

Multiple Comparisons

P-value is given by

Fmc=1-Fbp[0,-y2-a Abs[y1],a]

```
Fmc[y1_, y2_, a_] := 1 - Fbp[0, -y2 - a Abs[y1], a]
```

Then z-value is

```
Zmc[y1_, y2_, a_] := Qnorm[Fbp[0, -y2 - a Abs[y1], a]]
```

Next we consider a constant Cmc defined as

P_1(y2 > Cmc $- a$ Abs[y1] | (0, 0))=Fnorm[-z]

This is equivalent to

$1 - P_1$(y2 $\leq -a$ Abs[y1] | (0, $-$Cmc)) = Fnorm[$-z$]

```
Cmc[z_Real, a_] := Block[{u2cmc}, u2cmc /. FindRoot[Zbp[0, -u2cmc, a] == -z, {u2cmc, z}]]
```

Then the rejection probability is

Pmc = P_1(y2>c-a Abs[y1] | (μ1,μ2))

with c = Cmc

```
Pmc[μ1_, μ2_, a_, c_] := 1 - Fbp[μ1, μ2 - c, a]
```

■ Multiple Comparisons for k populations

In the above, only three populations case is treated. Here general k populations case is given.

probability of correct selection
P(CS) = P(y(k) ≥ y(i)-h, i=1,...,k-1)
we assume l.f.c.

```
PCSmc[k_, h_] := Module[{z}, NIntegrate[(Fnorm[z + h])^(k-1) dnorm[z], {z, -∞, ∞}]]
```

critical constant
solve h s.t. P(CS)=alpha

```
Hmc[k_, p_] := Module[{h}, h /. FindRoot[PCSmc[k, h] == p, {h, 0, √2 Qnorm[p]}]]
```

P-value is

```
Fmc[k_, h_] := 1 - PCSmc[k, h]
```

■ Somerville's Multiple Range Subset Selection

First consider the bootstrap probability for a modified hypothesis region

F2bp=Pr(y2 < -a Abs[y1] & d1 < Abs[y1] < d2 | μ1,μ2)

```
F2bp[μ1_, μ2_, a_, d1_, d2_] := Block[{y1f2bp},
  NIntegrate[((e^(-1/2 (y1f2bp-μ1)^2) + e^(-1/2 (y1f2bp+μ1)^2)) (1 - Erf[(a y1f2bp+μ2)/√2])) / (2 √(2 π)), {y1f2bp, d1, d2}]]
```

The rejection probability of the multiple range (MR) procedure is

Pmr=Pr(y2>c1 - a Abs[y1] & Abs[y1]>d2) + Pr(y2 > c2 - a Abs[y1] & Abs[y1]<d2)

```
Pmr[μ1_, μ2_, a_, c1_, c2_, d2_] :=
 1 - F2bp[μ1, μ2 - c1, a, d2, ∞] - F2bp[μ1, μ2 - c2, a, 0, d2]
```

The critical constants c1,c2,d2 in Pmr are defined below.

We focus on the "three normal populations" case:
$a0 = \frac{1}{\sqrt{3}}$, $d2 = \frac{\sqrt{3}}{2}$ c2, $c1 = \frac{2}{\sqrt{3}}$ z0, (z0 = -Qnorm[0.05]).

In below, a may be $\frac{1}{\sqrt{3}}$, or $-\frac{1}{\sqrt{3}}$.

Cmr=c2 is defined below.
Pmr[0,0,$\frac{1}{\sqrt{3}}$,$\frac{2}{\sqrt{3}}$ z0,c2,$\frac{\sqrt{3}}{2}$ c2]=Fnorm[-z0]

```
Cmr[z_ /; z > 0, a_] := Block[{u2cmr},
  u2cmr /. FindRoot[Pmr[0, 0, a, √(1 + a^2) z, u2cmr, u2cmr/(2 Abs[a])] == Fnorm[-z], {u2cmr, z}]]
```

Let Y2mr be y2 for rejection surface

```
Y2mr[y1_, a_, c1_, c2_, d2_] := If[Abs[y1] < d2, c2 - a Abs[y1], c1 - a Abs[y1]]
```

```
Y2mr[y1_, a_, c1_, c2_] := Y2mr[y1, a, c1, c2, c2/(2 Abs[a])]
```

The z-value is

```
Zmr[y1_, y2_, a_] :=
 Block[{xzmr}, xzmr /. FindRoot[zmrint0[xzmr, y1, y2, a], {xzmr, Zmc[y1, y2, a]}]]
```

where zmrint0 is a internal function defined by

```
zmrint0[uz_Real, y1_, y2_, a_] := Y2mr[y1, a, Sqrt[1 + a^2] uz, Cmr[uz, a]] - y2
```

The p-value is

```
Fmr[y1_, y2_, a_] := Fnorm[-Zmr[y1, y2, a]]
```

■ Rejection Region Plot (convex)

■ hypothesis

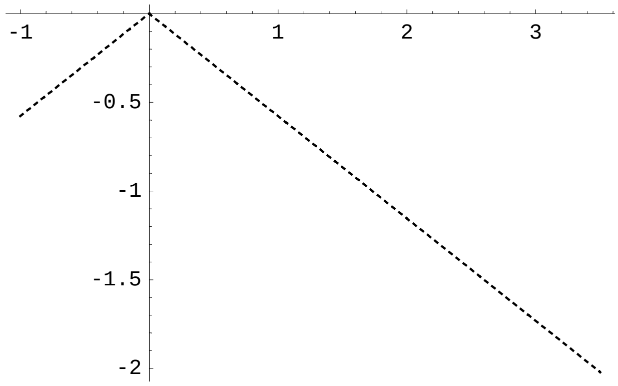
```
a0 = 1/Sqrt[3]; z0 = -Qnorm[0.05]; s0 = 1; x0 = 5;
```

```
{ymin, ymax} = {-2.2, 2.3};
```

```
xmin = -1; xmax = 3.5;
```

g0=hypothesis surface

```
g0 = Plot[-a0 Abs[x], {x, xmin, xmax},
  PlotStyle -> {Thickness[0.004], Dashing[{0.005, 0.01}]}]
```



- Graphics -

■ sample point-1

the three population example; x1 = $-\frac{5}{2}$; x2 = −1;
x3 = 0; The null hypothesis is H1 = mu1 is the largest.

```
xsp1 = -5 / 2; xsp2 = -1; ysp1 = -xsp2/Sqrt[2]; ysp2 = (-2 xsp1 + xsp2)/Sqrt[6];
```

```
{ysp1, ysp2}
```

$\left\{\frac{1}{\sqrt{2}}, 2\sqrt{\frac{2}{3}}\right\}$

```
{ysp1, ysp2} // N
```

```
{0.707107, 1.63299}
```

p-values of nt,mc,mr

```
{Fnt[ysp1, ysp2, a0] // N, Fmc[ysp1, ysp2, a0], Fmr[ysp1, ysp2, a0]}
```

```
{0.0385499, 0.0686348, 0.0790933}
```

p-values of mb(k=1,2,3,4)

```
Table[Fmb[k, s0][ysp1, ysp2, a0], {k, 1, 4}]
```

```
{0.0196601, 0.0459926, 0.0624965, 0.0796764}
```

```
gs1 = Graphics[{PointSize[0.03], Point[{ysp1, ysp2}]}]
```

```
- Graphics -
```

■ Another implementation of MC test

Calculate the mc-pvalue using another implementation for confirming Fmc above for point-1

The test statitic = 2.5, and m+2=3 populations case.

```
Fmc[3, 2.5]
```

```
0.0686348
```

In addition we calculate k populations case for k =10,100,1000,10000.

10 populatins case

```
Fmc[10, 2.5]
```

```
0.190831
```

100 populatins case

```
Fmc[100, 2.5]
```

0.498864

1000 populatins case

```
Fmc[1000, 2.5]
```

0.757308

10000 populatins case

```
Fmc[10000, 2.5]
```

0.90251

■ sample point-2

the three population example; x1 = $-\frac{5}{2}$; x2 = −9 / 2;
x3 = 0; The null hypothesis is H1 = mu1 is the largest.

xsp1 = −5 / 2; xsp2 = −9 / 2; ysp1 = $\frac{-xsp2}{\sqrt{2}}$**; ysp2 =** $\frac{-2\,xsp1 + xsp2}{Sqrt[6]}$**;**

```
{ysp1, ysp2}
```

$\left\{\frac{9}{2\sqrt{2}}, \frac{1}{2\sqrt{6}}\right\}$

```
{ysp1, ysp2} // N
```

{3.18198, 0.204124}

p-values of nt,mc,mr

```
{Fnt[ysp1, ysp2, a0] // N, Fmc[ysp1, ysp2, a0], Fmr[ysp1, ysp2, a0]}
```

{0.0385499, 0.0686348, 0.0385499}

p-values of mb(k=1,2,3,4)

```
Table[Fmb[k, s0][ysp1, ysp2, a0], {k, 1, 4}]
```

{0.0384623, 0.0393149, 0.0367451, 0.0383303}

```
gs2 = Graphics[{PointSize[0.03], Point[{ysp1, ysp2}]}]
```

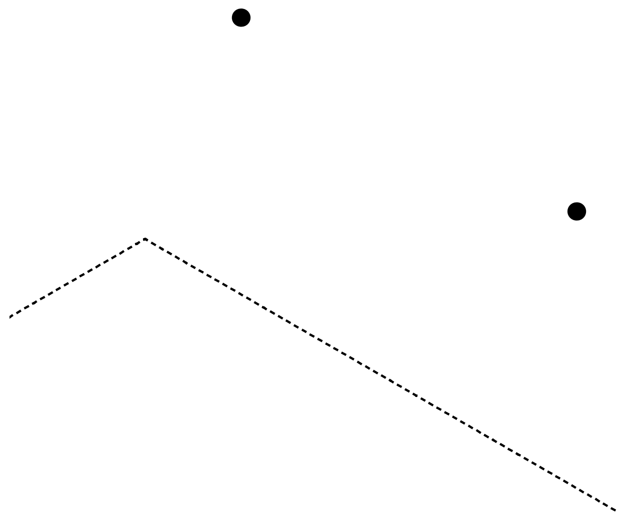
- Graphics -

```
Table[Fmb[k, 1][ysp1, ysp2, a0], {k, 1, 4}]
```

{0.0384623, 0.0393149, 0.0367451, 0.0383303}

■ Show hypothesis, point-1, point-2

```
Show[{g0, gs1, gs2}, Axes → False,
 PlotRange → {{xmin, xmax}, {ymin, ymax}}, AspectRatio → (ymax - ymin)/(xmax - xmin)]
```



- Graphics -

■ nt, mc, mr

c−values (vertical distance) = $\sqrt{\frac{2}{3}}\,\delta$

$\left\{z0\sqrt{1+a0^2},\ Cmc[z0,\ a0],\ Cmr[z0,\ a0]\right\}$

{1.89931, 2.21279, 2.2733}

Distances to hypothesis

$\left\{z0\sqrt{1+a0^2},\ Cmc[z0,\ a0],\ Cmr[z0,\ a0]\right\} \Big/ \sqrt{1+a0^2}$

{1.64485, 1.91633, 1.96873}

Critical constants

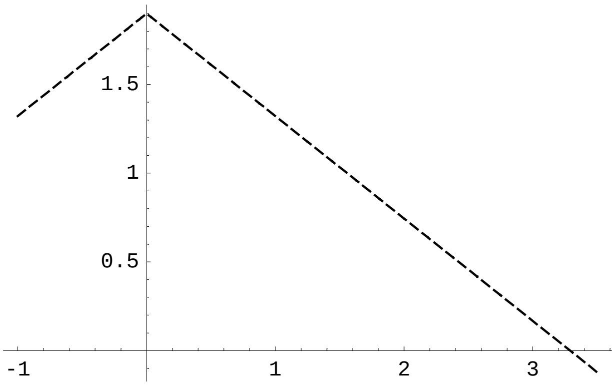
$c1 = z0\sqrt{1+a0^2};\ c2 = Cmr[z0,\ a0];\ z2 = c2 \Big/ \sqrt{1+a0^2};$

{c1, c2, z2}

{1.89931, 2.2733, 1.96873}

gn=normal test

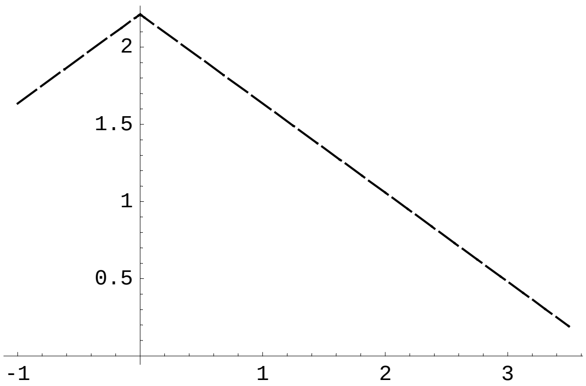
```
gn = Plot[-a0 Abs[x] + z0 Sqrt[1 + a0^2], {x, xmin, xmax},
  PlotStyle → {Thickness[0.004], Dashing[{0.015, 0.01}]}]
```



- Graphics -

gm=multiple comparisons

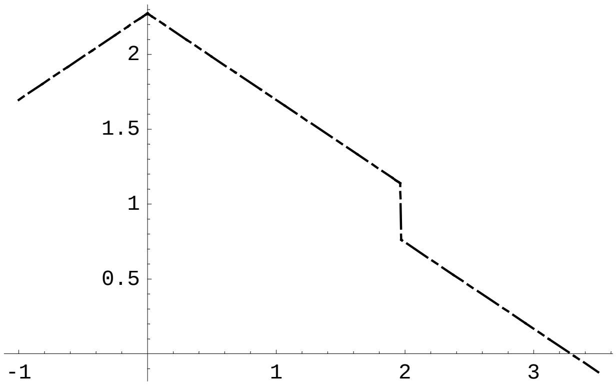
```
gm = Plot[-a0 Abs[x] + Cmc[z0, a0], {x, xmin, xmax},
  PlotStyle → {Thickness[0.004], Dashing[{0.04, 0.01}]}]
```



- Graphics -

gr=multiple range

```
gr = Plot[Y2mr[x, a0, c1, c2, z2], {x, xmin, xmax},
  PlotStyle → {Thickness[0.004], Dashing[{0.01, 0.01, 0.04, 0.01}]}]
```



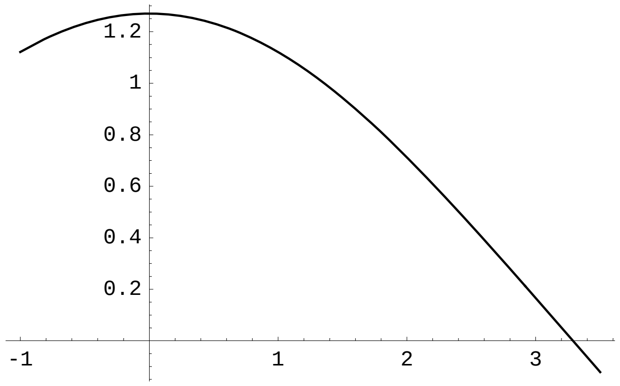
- Graphics -

■ mb

g1,g2,g3,g4=multiscale bootstrap (this takes a few minitus)

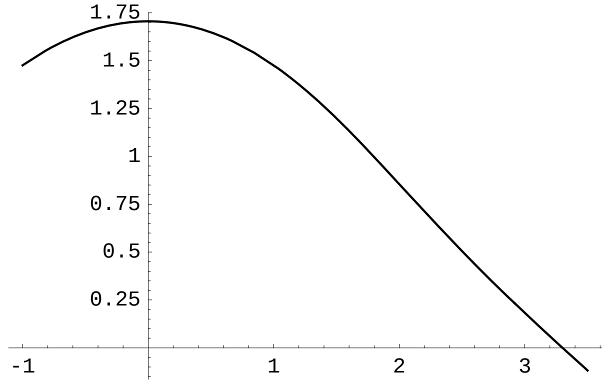
```
rmb[x_, k_] := -Rmb[k, s0][Abs[x], z0, a0, x0]
```

```
g1 = Plot[rmb[x, 1], {x, xmin, xmax}, PlotStyle → {Thickness[0.004]}]
```



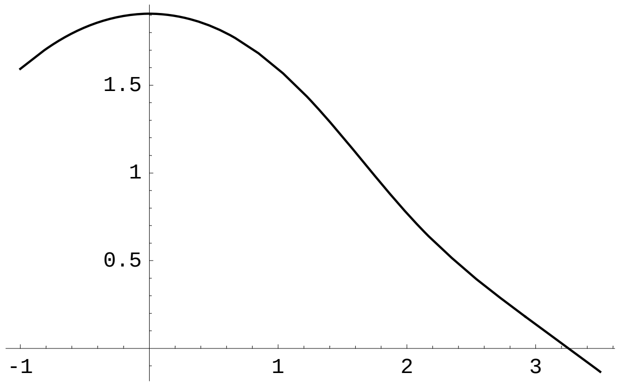
- Graphics -

```
g2 = Plot[rmb[x, 2], {x, xmin, xmax}, PlotStyle → {Thickness[0.004]}]
```



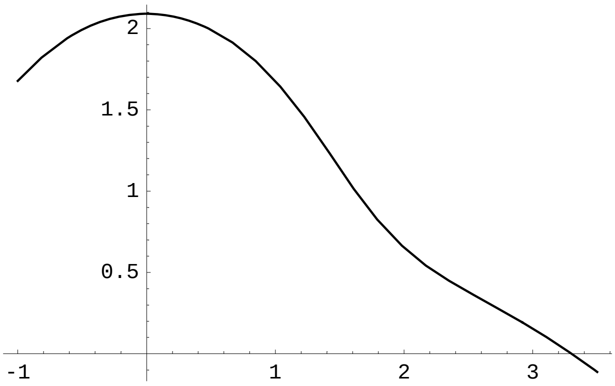
- Graphics -

```
g3 = Plot[rmb[x, 3], {x, xmin, xmax}, PlotStyle → {Thickness[0.004]}]
```



- Graphics -

```
g4 = Plot[rmb[x, 4], {x, xmin, xmax}, PlotStyle → {Thickness[0.004]}]
```



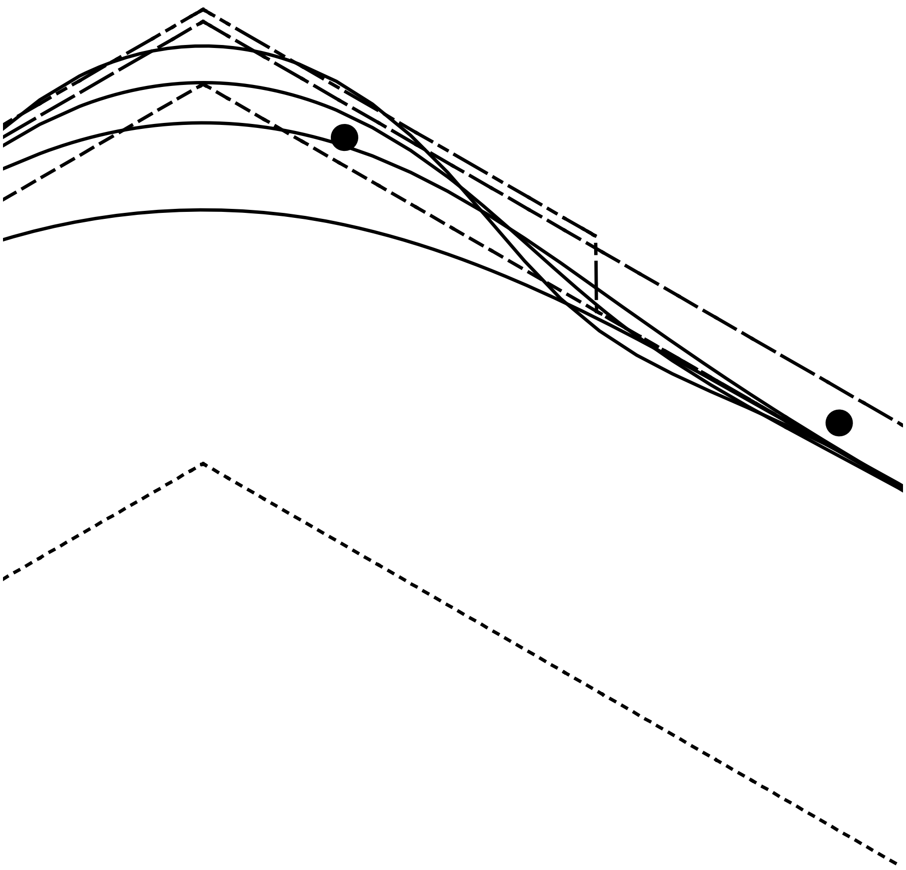
- Graphics -

■ save

```
gconv = {g0, gs1, gs2, gn, gm, gr, g1, g2, g3, g4};
```

```
Save["tech200605-gconv.txt", gconv]
```

```
g = Show[gconv, Axes → False,
  PlotRange → {{xmin, xmax}, {ymin, ymax}}, AspectRatio → (ymax - ymin)/(xmax - xmin)]
```



- Graphics -

■ Rejection Region Plot (concave)

■ hypothesis

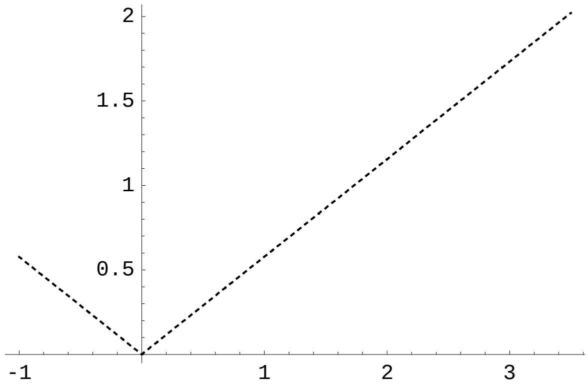
```
a0 = -1/√3; z0 = -Qnorm[0.05]; s0 = 1; x0 = 5;
```

```
{ymin, ymax} = {-0.2, 4.3};
```

```
xmin = -1; xmax = 3.5;
```

g0=hypothesis surface

```
g0 = Plot[-a0 Abs[x], {x, xmin, xmax},
  PlotStyle → {Thickness[0.004], Dashing[{0.005, 0.01}]}]
```




```
- Graphics -
```

■ sample point-3

```
the three population example;  x1 = 2; x2 = -1; x3 = 0;
The null hypothesis is H1 = mu1 is not the largest.
    Below,  the sign of ysp2 is changed from the convex case.
```

```
xsp1 = 2; xsp2 = -1; ysp1 = -xsp2/√2; ysp2 = -(-2 xsp1 + xsp2)/Sqrt[6];
```

```
{ysp1, ysp2}
```

$\left\{\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{6}}\right\}$

```
{ysp1, ysp2} // N
```

```
{0.707107, 2.04124}
```

p-values of nt,mc,mr

```
{Fnt[ysp1, ysp2, a0] // N, Fmc[ysp1, ysp2, a0], Fmr[ysp1, ysp2, a0]}
```

```
{0.0786496, 0.0230664, 0.0225127}
```

p-values of mb(k=1,2,3,4)

```
Table[Fmb[k, s0][ysp1, ysp2, a0], {k, 1, 4}]
```

{0.0884579, 0.0553918, 0.0625131, 0.0906105}

```
gs1 = Graphics[{PointSize[0.03], Point[{ysp1, ysp2}]}]
```

- Graphics -

■ sample point-4

the three population example; x1 = 2; x2 = −9 / 2; x3 = 0;
The null hypothesis is H1 = mu1 is not the largest.
Below, the sign of ysp2 is changed from the convex case.

$$xsp1 = 2;\ xsp2 = -9/2;\ ysp1 = \frac{-xsp2}{\sqrt{2}};\ ysp2 = -\frac{-2\,xsp1 + xsp2}{Sqrt[6]};$$

```
{ysp1, ysp2}
```

$$\left\{\frac{9}{2\sqrt{2}},\ \frac{17}{2\sqrt{6}}\right\}$$

```
{ysp1, ysp2} // N
```

{3.18198, 3.47011}

p-values of nt,mc,mr

```
{Fnt[ysp1, ysp2, a0] // N, Fmc[ysp1, ysp2, a0], Fmr[ysp1, ysp2, a0]}
```

{0.0786496, 0.0230664, 0.0786496}

p-values of mb(k=1,2,3,4)

```
Table[Fmb[k, s0][ysp1, ysp2, a0], {k, 1, 4}]
```

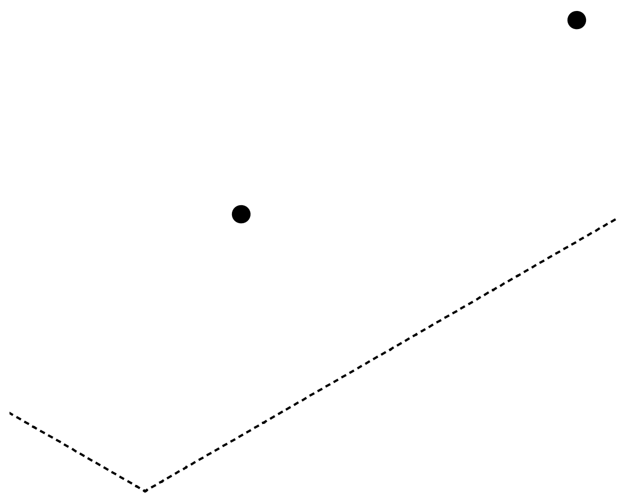
{0.0786499, 0.0786436, 0.0787025, 0.0784167}

```
gs2 = Graphics[{PointSize[0.03], Point[{ysp1, ysp2}]}]
```

- Graphics -

■ Show hypothesis, point-3, point-4

```
Show[{g0, gs1, gs2}, Axes → False,
 PlotRange → {{xmin, xmax}, {ymin, ymax}}, AspectRatio → (ymax - ymin)/(xmax - xmin)]
```



- Graphics -

■ nt, mc, mr

c–values (vertical distance) = $\sqrt{\frac{2}{3}}\,\delta$

$\{z0\sqrt{1+a0^2},\ \mathrm{Cmc}[z0, a0],\ \mathrm{Cmr}[z0, a0]\}$

{1.89931, 1.27007, 1.22596}

Distances to hypothesis

$\{z0\sqrt{1+a0^2},\ \mathrm{Cmc}[z0, a0],\ \mathrm{Cmr}[z0, a0]\} \Big/ \sqrt{1+a0^2}$

{1.64485, 1.09992, 1.06172}

Critical constants

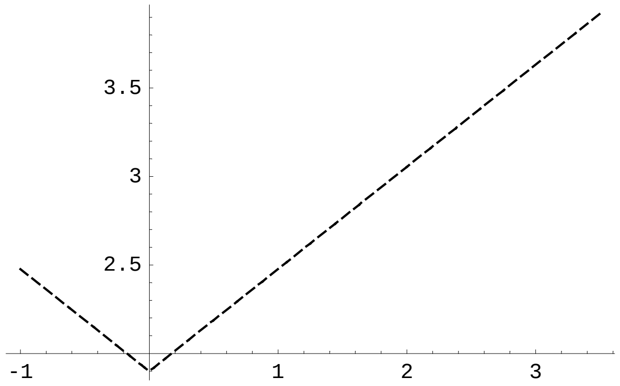
$c1 = z0\sqrt{1+a0^2};\ c2 = \mathrm{Cmr}[z0, a0];\ z2 = c2 \Big/ \sqrt{1+a0^2};$

{c1, c2, z2}

{1.89931, 1.22596, 1.06172}

gn=normal test

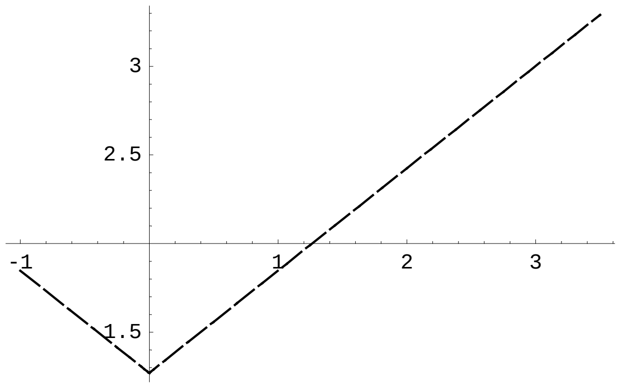
```
gn = Plot[-a0 Abs[x] + z0 Sqrt[1 + a0^2], {x, xmin, xmax},
  PlotStyle → {Thickness[0.004], Dashing[{0.015, 0.01}]}]
```



- Graphics -

gm=multiple comparisons

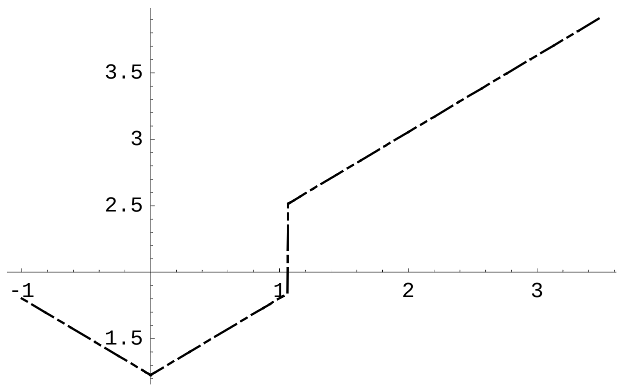
```
gm = Plot[-a0 Abs[x] + Cmc[z0, a0], {x, xmin, xmax},
  PlotStyle → {Thickness[0.004], Dashing[{0.04, 0.01}]}]
```



- Graphics -

gr=multiple range

```
gr = Plot[Y2mr[x, a0, c1, c2, z2], {x, xmin, xmax},
  PlotStyle → {Thickness[0.004], Dashing[{0.01, 0.01, 0.04, 0.01}]}]
```



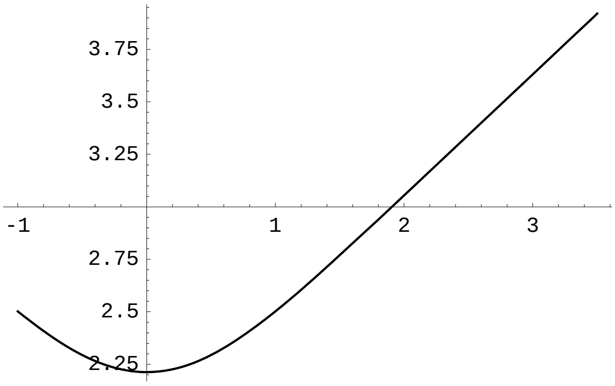
- Graphics -

■ mb

g1,g2,g3,g4=multiscale bootstrap (this takes a few minitus)

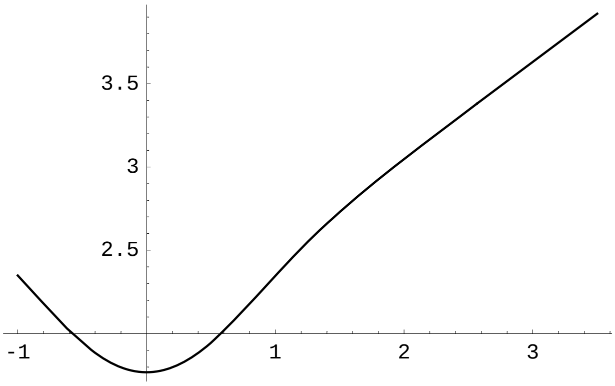
```
rmb[x_, k_] := -Rmb[k, s0][Abs[x], z0, a0, x0]
```

```
g1 = Plot[rmb[x, 1], {x, xmin, xmax}, PlotStyle → {Thickness[0.004]}]
```



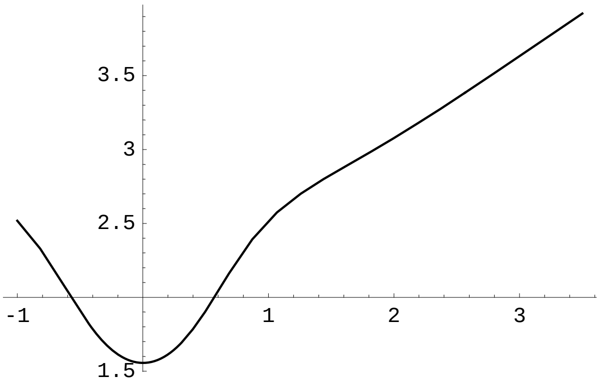
- Graphics -

```
g2 = Plot[rmb[x, 2], {x, xmin, xmax}, PlotStyle → {Thickness[0.004]}]
```



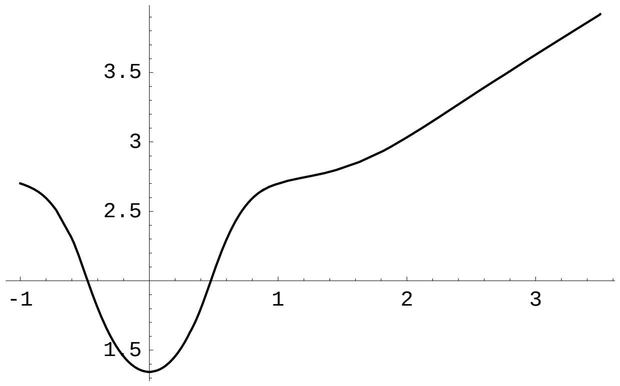
- Graphics -

```
g3 = Plot[rmb[x, 3], {x, xmin, xmax}, PlotStyle → {Thickness[0.004]}]
```



- Graphics -

```
g4 = Plot[rmb[x, 4], {x, xmin, xmax}, PlotStyle → {Thickness[0.004]}]
```



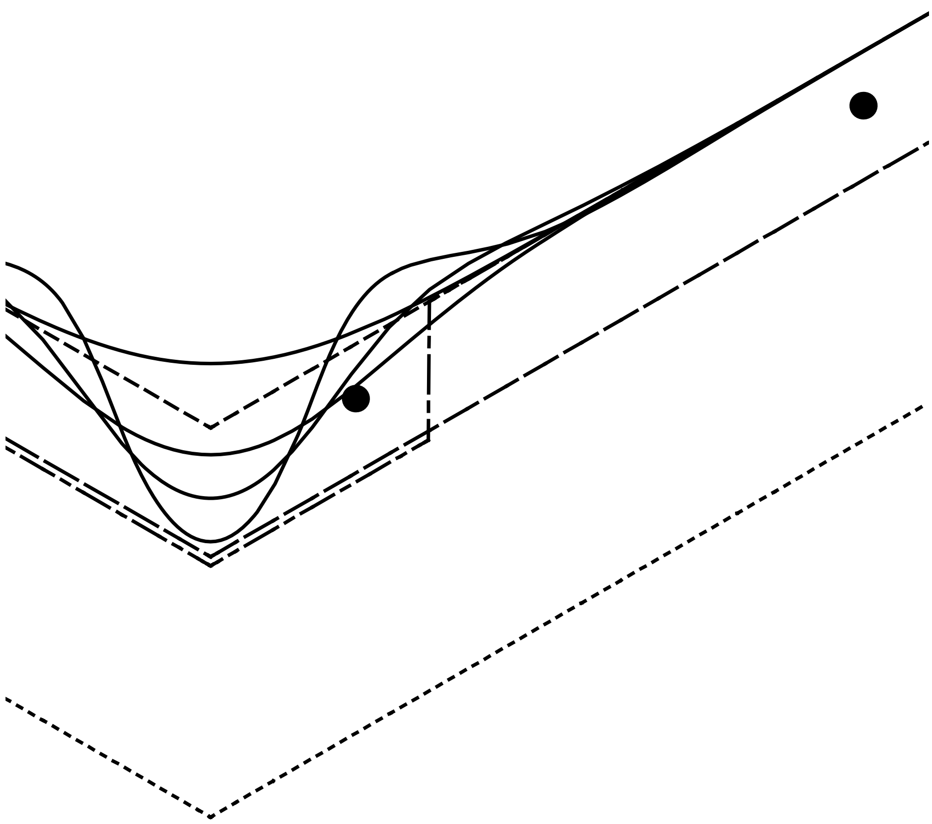
- Graphics -

■ save

```
gconc = {g0, gs1, gs2, gn, gm, gr, g1, g2, g3, g4};
```

```
Save["tech200605-gconc.txt", gconc]
```

```
g = Show[gconc, Axes → False,
  PlotRange → {{xmin, xmax}, {ymin, ymax}}, AspectRatio → (ymax - ymin)/(xmax - xmin)]
```



- Graphics -

Rejection Probability Table (convex)

parameter

```
a0 = 1/Sqrt[3];

z0 = -Qnorm[0.05]; dmax = 8; s0 = 1; x0 = 5;

xlist = Table[d/Sqrt[2], {d, 0, dmax}];
```

mb (taks 25mins)

nt

mc

mr

save

```
pconv = {pnt, pmc, pmr, pmb1, pmb2, pmb3, pmb4};

pconv // TableForm
```

0.0878106	0.0559223	0.0505433	0.0500283	0.0500008	0.05
0.05	0.0306891	0.027902	0.0276723	0.0276617	0.0276614
0.05	0.0359879	0.0413653	0.0471165	0.0494521	0.0499423
0.133921	0.0777059	0.0579416	0.0519658	0.0504033	0.0500655
0.0765524	0.0473841	0.0442656	0.0467426	0.0488128	0.0497
0.0660885	0.0451039	0.0469624	0.0501802	0.0508711	0.0504612
0.0621694	0.0460812	0.0499031	0.0517078	0.050772	0.0500083

```
round[x_] = NumberForm[x * 100, 3];

Map[round, pconv, {2}] // TableForm
```

8.78	5.59	5.05	5.	5.	5.	5.	5.	5.
5.	3.07	2.79	2.77	2.77	2.77	2.77	2.77	2.77
5.	3.6	4.14	4.71	4.95	4.99	5.	5.	5.
13.4	7.77	5.79	5.2	5.04	5.01	5.	5.	5.
7.66	4.74	4.43	4.67	4.88	4.97	4.99	5.	5.
6.61	4.51	4.7	5.02	5.09	5.05	5.01	5.	5.
6.22	4.61	4.99	5.17	5.08	5.	4.99	5.	5.

```
Save["tech200605-pconv.txt", pconv]
```

■ Rejection Probability Table (concave)

■ parameter

```
a0 = -1/Sqrt[3];

z0 = -Qnorm[0.05]; dmax = 8; s0 = 1; x0 = 5;

xlist = Table[d/Sqrt[2], {d, 0, dmax}];
```

■ mb (taks 25mins)

■ nt

■ mc

■ mr

■ save

```
pconc = {pnt, pmc, pmr, pmb1, pmb2, pmb3, pmb4};

pconc // TableForm
```

0.0121894	0.0270621	0.04085	0.0478027	0.0496991	0.0499772
0.05	0.0917547	0.12144	0.132966	0.135392	0.135667
0.05	0.0857258	0.0949549	0.0775088	0.0592358	0.0517185
0.00844598	0.020419	0.0343131	0.0441504	0.0484996	0.0497347
0.0195445	0.039935	0.0544011	0.0566495	0.053375	0.0509713
0.0237285	0.0448911	0.0551778	0.0529767	0.0496021	0.0492008
0.0266674	0.0470522	0.053737	0.0503377	0.048924	0.0497796

```
round[x_] = NumberForm[x * 100, 3];

Map[round, pconc, {2}] // TableForm
```

1.22	2.71	4.09	4.78	4.97	5.	5.	5.	5.
5.	9.18	12.1	13.3	13.5	13.6	13.6	13.6	13.6
5.	8.57	9.5	7.75	5.92	5.17	5.02	5.	5.
0.845	2.04	3.43	4.42	4.85	4.97	5.	5.	5.
1.95	3.99	5.44	5.66	5.34	5.1	5.02	5.	5.
2.37	4.49	5.52	5.3	4.96	4.92	4.97	4.99	5.
2.67	4.71	5.37	5.03	4.89	4.98	5.01	5.01	5.

```
Save["tech200605-pconc.txt", pconc]
```

Filter Representations

Multiscale Bootstrap

Rejection Surface

By substituting

$$\psi(\sigma^2) = v + E_\sigma(h(U^*) \mid u)$$

into the definition of multiscale bootstrap, we get

$$\hat{z}_{k,\sigma 0} = \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \left(\frac{d^j \psi(x)}{d x^j}\right)_{\sigma 0^2} = v + \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \left(\frac{d^j E_\sigma(h(U^*) \mid u)}{d(\sigma^2)^j}\right)_{\sigma 0^2}$$

Thus the rejection surface is

$$r(u)+z = \hat{z}(u, v) - v = \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \left(\frac{d^j E_\sigma(h(U^*) \mid u)}{d(\sigma^2)^j}\right)_{\sigma 0^2}$$

Unbiased Surface

For normal density, we have

$$\psi(\sigma^2) = v + \int e^{i\omega u - \sigma^2 \frac{\omega^2}{2}} H(\omega)\, d\omega$$

$$\left(\frac{d^j \psi(x)}{d x^j}\right)_{\sigma 0^2} = \int e^{i\omega u - \sigma 0^2 \frac{\omega^2}{2}} \left(-\frac{\omega^2}{2}\right)^j H(\omega)\, d\omega \quad \text{for } j \geq 1$$

Thus the rejection filter is

$$e^{\frac{\omega^2}{2}} (1 - J_{k,\sigma 0}(\omega)) = \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} e^{-\sigma 0^2 \frac{\omega^2}{2}} \left(-\frac{\omega^2}{2}\right)^j$$

The unbiased filter is obtained by multiplying $e^{-\frac{\omega^2}{2}}$ to above.

$$1 - J_{k,\sigma 0}(\omega) = \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} e^{-(1+\sigma 0^2)\frac{\omega^2}{2}} \left(-\frac{\omega^2}{2}\right)^j$$

The unbiased surface is then

$$s(u) = \int e^{i\omega u} (1 - J_{k,\sigma 0}(\omega)) H(\omega)\, d\omega = \int e^{i\omega u} \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} e^{-(1+\sigma 0^2)\frac{\omega^2}{2}} \left(-\frac{\omega^2}{2}\right)^j H(\omega)\, d\omega$$

$$= \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \int e^{i \omega u} e^{-(1+\sigma 0^2) \frac{\omega^2}{2}} \left(-\frac{\omega^2}{2}\right)^j H(\omega)\, d\omega$$

$$= -V + \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \left(\frac{d^j \psi(x)}{d x^j}\right)_{1+\sigma 0^2}$$

$$= \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \left(\frac{d^j E_\sigma(h(U^*) \mid u)}{d(\sigma^2)^j}\right)_{1+\sigma 0^2}$$

■ Rejection filter

From the definition, the rejection filter is given by

$$\text{RFmb}[k\_, \sigma 0\_][\omega\_] := \sum_{j=0}^{k-1} \frac{(1+\sigma 0^2)^j \omega^{2j}}{2^j j!} e^{-\sigma 0^2 \frac{\omega^2}{2}}$$

It simplifies to

```
Simplify[RFmb[k, σ0][ω]]
```

$$\frac{e^{\frac{\omega^2}{2}} \text{Gamma}[k, \frac{1}{2}(1+\sigma 0^2)\omega^2]}{\text{Gamma}[k]}$$

■ J function

The J function is obtained from the rejection filter

$$\text{Jmb}[k\_, \sigma 0\_][\omega\_] = \text{Simplify}\left[1 - e^{-\frac{\omega^2}{2}} \text{RFmb}[k, \sigma 0][\omega]\right]$$

$$1 - \frac{\text{Gamma}[k, \frac{1}{2}(1+\sigma 0^2)\omega^2]}{\text{Gamma}[k]}$$

Using the standard notation of the incomplete gamma function, it is written as foo below

$$\text{gamma}[k\_, z\_] = \text{Simplify}\left[\int_0^z t^{k-1} e^{-t}\, dt,\ k > 0\right]$$

```
Gamma[k] - Gamma[k, z]
```

$$\text{foo} := \frac{\text{gamma}[k, \frac{1}{2}(1+\sigma 0^2)\omega^2]}{\text{Gamma}[k]}$$

```
Simplify[Jmb[k, σ0][ω] - foo]
```

```
0
```

■ Another J expression

Yet another expression for the J function

```
foo := Sum[((-1)^(j-k) (1 + σ0^2)^j ω^(2 j))/((k - 1)! (j - k)! j 2^j), {j, k, ∞}]
```

$$\text{foo} := \sum_{j=k}^{\infty} \frac{(-1)^{j-k} (1+\sigma 0^2)^j \omega^{2j}}{(k-1)!\,(j-k)!\, j\, 2^j}$$

```
FullSimplify[Jmb[k, σ0][ω] - foo, k ∈ Integers]
```

0

■ Bootstrap Iteration

■ Iteration formula

$D_1(\omega) = H(\omega) F(\omega)$

$D_{k+1}(\omega) = (1 - F(\omega)) D_k(\omega) + H(\omega)$ for $k \geq 1$

By induction, we get for $D_k(\omega)$

$$\text{foo1} := H[\omega] \frac{1 - (1 + F[\omega])(1 - F[\omega])^k}{F[\omega]}$$

This is confirmed as follows

```
Simplify[foo1 /. k → 1]
```

F[ω] H[ω]

```
Simplify[(foo1 /. k → k + 1) - (1 - F[ω]) foo1]
```

H[ω]

Another expression of $D_k(\omega)$ is

$$\text{foo2} := H[\omega] \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!} (-F[\omega])^j$$

```
FullSimplify[foo2 - foo1]
```

0

■ J function

From the definition, the J function should correspond to

$(1 + F[\omega])(1 - F[\omega])^k$

For the normal model, it becomes

$$\text{Jbi}[k\_][\omega\_] := \left(1 + e^{-\frac{\omega^2}{2}}\right) \left(1 - e^{-\frac{\omega^2}{2}}\right)^k$$

■ Brief summary on Stirling numbers of second kind

The noncentral Stirling number of second kind is

```
S2[j_, k_, r_] := 1/k! Sum[(-1)^(k-i) k!/(i! (k-i)!) (i+r)^j, {i, 0, k}]
```

$$S2[j\_, k\_, r\_] := \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!\,(k-i)!}(i+r)^j$$

See Charalambides, Charalambos A. (2005) "Combinatorial methods in discrete distributions" Wiley.

The (central) Stirling number of second kind is (equivalent to StirlingS2[j,k])

```
S2[j_, k_] := S2[j, k, 0]
```

As seen in the table, S2[j,k,r]=0 for k>j

```
Table[Simplify[S2[j, k, r]], {j, 1, 5}, {k, 1, 5}] // TableForm
```

1	0	0	0	0
$1+2r$	1	0	0	0
$1+3r+3r^2$	$3(1+r)$	1	0	0
$-r^4+(1+r)^4$	$7+12r+6r^2$	$6+4r$	1	0
$-r^5+(1+r)^5$	$5(3+7r+6r^2+2r^3)$	$5(5+6r+2r^2)$	$5(2+r)$	1

```
Table[S2[j, k], {j, 1, 5}, {k, 1, 5}] // TableForm
```

1	0	0	0	0
1	1	0	0	0
1	3	1	0	0
1	7	6	1	0
1	15	25	10	1

The exponential generating function is foo1, which gives foo1=foo2 below.

$$\text{foo1} := \frac{e^{r x}(e^x-1)^k}{k!}$$

$$\text{foo2} := \sum_{j=k}^{\infty} S2[j, k, r]\,\frac{x^j}{j!}$$

The identity foo1=foo2 is verified for k=4 up to the first five terms as follows

```
FullSimplify[
 Table[S2[t + k, k, r] - (t + k)! SeriesCoefficient[Series[foo1/x^k, {x, 0, 5}], t],
   {t, 0, 4}] /. k -> 4]
```

{0, 0, 0, 0, 0}

The vertical recurrence relation (only for r=0)

$$S2[n+1, k+1] = \sum_{j=k}^{n}\frac{n!}{j!\,(n-j)!}\,S2[j, k] = S2[n, k, 1]$$

```
Table[Expand[S2[n + 1, k + 1] - S2[n, k, 1]], {n, 0, 5}, {k, 0, 5}] // TableForm
```

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

■ Another J expression

Let $x = -\frac{\omega^2}{2}$ here.

Apply the generating function $e^{r x}\,(e^x - 1)^k = k!\sum_{j=k}^{\infty} S2[j, k, r]\,\frac{x^j}{j!}$;

$r = 0$ gives $(e^x - 1)^k = k!\sum_{j=k}^{\infty} S2[j, k]\,\frac{x^j}{j!}$

$r = 1$ gives $e^x\,(e^x - 1)^k = k!\sum_{j=k}^{\infty} S2[j, k, 1]\,\frac{x^j}{j!} = k!\sum_{j=k}^{\infty} S2[j + 1, k + 1]\,\frac{x^j}{j!}$

Thus, $(1 + e^x)\,(e^x - 1)^k = k!\sum_{j=k}^{\infty} (S2[j, k] + S2[j + 1, k + 1])\,\frac{x^j}{j!}$

Therefore, the J function is

$$\left(1 + e^{-\frac{\omega^2}{2}}\right)\left(1 - e^{-\frac{\omega^2}{2}}\right)^k = (-1)^k\,k!\sum_{j=k}^{\infty} (S2[j, k] + S2[j + 1, k + 1])\,\frac{1}{j!}\left(-\frac{\omega^2}{2}\right)^j$$

■ Rejection Surface

From the expression of $D_k(\omega)$, we have

$$\hat{z}_{k,bi} = v + \int e^{i\,\omega\,u}\,H[\omega]\sum_{j=0}^{k}\frac{k!\,(k - 1 - 2 j)}{(j + 1)!\,(k - j)!}\,(-1)^j\,e^{-j\,\frac{\omega^2}{2}}\,d\omega$$

Thus the rejection surface for bootstrap iteration is

$$r(u)+z = \hat{z} - v = \sum_{j=0}^{k}\frac{k!\,(k-1-2 j)}{(j+1)!\,(k-j)!}\,(-1)^j\int e^{i\,\omega\,u}\,H[\omega]\,e^{-j\,\frac{\omega^2}{2}}\,d\omega$$

$$= \sum_{j=0}^{k}\frac{k!\,(k - 1 - 2 j)}{(j + 1)!\,(k - j)!}\,(-1)^j\int e^{i\,\omega\,u}\,H[\omega]\,e^{-j\,\frac{\omega^2}{2}}\,d\omega$$

$$= \sum_{j=0}^{k}\frac{k!\,(k - 1 - 2 j)}{(j + 1)!\,(k - j)!}\,(-1)^j\,E_{\sqrt{j}}\,(h\,(U^*)\mid u)$$

■ Unbiased Surface

By multiplying $e^{-\frac{\omega^2}{2}}$ to the filter in the rejection surface expression, we have

$$s(u) = \int e^{i\omega u}\, H[\omega]\, e^{-\frac{\omega^2}{2}} \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!}\,(-1)^j\, e^{-j\frac{\omega^2}{2}}\, d\omega$$

$$= \int e^{i\omega u}\, H[\omega] \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!}\,(-1)^j\, e^{-(1+j)\frac{\omega^2}{2}}\, d\omega$$

$$= \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!}\,(-1)^j\, E_{\sqrt{1+j}}\,(h\,(U^*)\,\mid u)$$

Surfaces of Revolution Example

■ A simple example of surfaces of revolution

■ Expected values

We work on the hypothesis surface defined by $h(u) = ||u||^q$

First we give the expected value of chisqaure random variable $\chi_m{}^2$. The expected value of $(\chi_m{}^2)^{\frac{q}{2}} = \chi_m{}^q$ is given by

$$\text{Echisq}[m\_,\, q\_] := \frac{2^{\frac{q}{2}}\,\text{Gamma}[\frac{m+q}{2}]}{\text{Gamma}[\frac{m}{2}]}$$

For the noncentral chisquare random variable $\chi_m{}^2(\delta^2)$ with noncentrality δ^2, $E\left((\chi_m{}^2(\delta^2))^{q/2}\right)$ is given by

$$E\left((\chi_m{}^2(\delta^2))^{q/2}\right) = \sum_{j=0}^{\infty} \frac{(\delta^2/2)^j}{j!}\, e^{-\delta^2/2}\, E(\chi_{m+2j}{}^q)$$

$$\text{Simplify}\left[\sum_{j=0}^{\infty} \frac{(\delta^2/2)^j}{j!}\, e^{-\delta^2/2}\, \text{Echisq}[m+2j,\, q]\right]$$

$$\frac{2^{q/2}\, e^{-\frac{\delta^2}{2}}\, \text{Gamma}[\frac{m+q}{2}]\, \text{Hypergeometric1F1}[\frac{m+q}{2},\, \frac{m}{2},\, \frac{\delta^2}{2}]}{\text{Gamma}[\frac{m}{2}]}$$

$$= E(\chi_m{}^q)\, e^{-\delta^2/2}\; {}_1F_1\left(\frac{m+q}{2},\, \frac{m}{2},\, \frac{\delta^2}{2}\right)$$

$= E(\chi_m{}^q)\; {}_1F_1\left(-\frac{q}{2},\, \frac{m}{2},\, -\frac{\delta^2}{2}\right)$; The last equality follows from the identity $e^z\, {}_1F_1[a, b, -z] = {}_1F_1[b-a, b, z]$,

where $z = -\frac{\delta^2}{2}$, $a = \frac{m+q}{2}$, $b = \frac{m}{2}$.

$$\text{Echisq}[m\_,\, q\_,\, \delta\_] := \text{Echisq}[m,\, q]\, \text{Hypergeometric1F1}\left[-\frac{q}{2},\, \frac{m}{2},\, -\frac{\delta^2}{2}\right]$$

Therefore, $E_\sigma(\| U^* \|^q \mid u) = \sigma^q E(\chi_m{}^q(\| u \|^2 / \sigma^2))$ is given by

```
Euq[σ_, u_][m_, q_] := σ^q Echisq[m, q, u/σ]
```

```
Euq[σ, u][m, q]
```

$$\frac{2^{q/2} \sigma^q \mathrm{Gamma}\left[\frac{m+q}{2}\right] \mathrm{Hypergeometric1F1}\left[-\frac{q}{2}, \frac{m}{2}, -\frac{u^2}{2\sigma^2}\right]}{\mathrm{Gamma}\left[\frac{m}{2}\right]}$$

For σ=0, $E_0(\| U^* \|^q \mid u) := \| u \|^q$

```
Euq[0, u_][m_, q_] := (u^2)^(q/2)
```

■ Derivatives of the expected value

$$\mathrm{Hypergeometric1F1}\left[-\frac{q}{2}, \frac{m}{2}, -\frac{u^2}{2\sigma^2}\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\left(-\frac{q}{2}\right)_k}{\left(\frac{m}{2}\right)_k} \left(-\frac{u^2}{2\sigma^2}\right)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(-1)^k \mathrm{Gamma}\left[\frac{q}{2}+1\right]}{\mathrm{Gamma}\left[\frac{q}{2}+1-k\right]} \frac{\mathrm{Gamma}\left[\frac{m}{2}\right]}{\mathrm{Gamma}\left[\frac{m}{2}+k\right]} \left(-\frac{u^2}{2\sigma^2}\right)^k$$

Substitute this expression into Euq function, we have

$$\mathrm{Euq}[\sigma, u][m, q] = \frac{2^{q/2} \sigma^q \mathrm{Gamma}\left[\frac{m+q}{2}\right] \mathrm{Hypergeometric1F1}\left[-\frac{q}{2}, \frac{m}{2}, -\frac{u^2}{2\sigma^2}\right]}{\mathrm{Gamma}\left[\frac{m}{2}\right]}$$

$$= \sum_{k=0}^{\infty} \frac{2^{q/2} \sigma^q \mathrm{Gamma}\left[\frac{m+q}{2}\right]}{\mathrm{Gamma}\left[\frac{m}{2}\right]} \frac{1}{k!} \frac{(-1)^k \mathrm{Gamma}\left[\frac{q}{2}+1\right]}{\mathrm{Gamma}\left[\frac{q}{2}+1-k\right]} \frac{\mathrm{Gamma}\left[\frac{m}{2}\right]}{\mathrm{Gamma}\left[\frac{m}{2}+k\right]} \left(-\frac{u^2}{2\sigma^2}\right)^k$$

The each term in the summation is simplified to

```
foo1 = FullSimplify[PowerExpand[
   (2^(q/2) σ^q Gamma[(m+q)/2])/Gamma[m/2] (1/k!) ((-1)^k Gamma[q/2 + 1])/Gamma[q/2 + 1 - k] Gamma[m/2]/Gamma[m/2 + k] (-u^2/(2 σ^2))^k], k ∈ Integers]
```

$$\frac{2^{\frac{1}{2}(-2k+q)} u^{2k} \sigma^{-2k+q} \mathrm{Gamma}\left[1+\frac{q}{2}\right] \mathrm{Gamma}\left[\frac{m+q}{2}\right]}{k!\, \mathrm{Gamma}\left[k+\frac{m}{2}\right] \mathrm{Gamma}\left[1-k+\frac{q}{2}\right]}$$

We consider differentiation of σ^{-2k+q} with respect to σ^2. Note that

$$D[s^p, \{s, j\}] = p(p-1)\ldots(p-j+1)\, s^{p-j} = \frac{\mathrm{Gamma}[p+1]}{\mathrm{Gamma}[p+1-j]} s^{p-j}$$

Put $p = \frac{1}{2}(-2k+q)$, we have

$$D\left[s^{\frac{1}{2}(-2k+q)}, \{s, j\}\right] = \frac{\mathrm{Gamma}\left[\frac{1}{2}(-2k+q)+1\right]}{\mathrm{Gamma}\left[\frac{1}{2}(-2k+q)+1-j\right]} s^{\frac{1}{2}(-2k+q)-j}$$

Substituting σ^2 for s above, and putting it into foo1, we obtain

D[foo1, {σ^2, j}] =

$$\texttt{foo2} = \texttt{FullSimplify}\Big[\texttt{PowerExpand}\Big[\frac{2^{\frac{1}{2}(-2k+q)}\, u^{2k}\, \texttt{Gamma}[1+\frac{q}{2}]\, \texttt{Gamma}[\frac{m+q}{2}]}{k!\, \texttt{Gamma}[k+\frac{m}{2}]\, \texttt{Gamma}[1-k+\frac{q}{2}]} \frac{\texttt{Gamma}[\frac{1}{2}(-2k+q)+1]}{\texttt{Gamma}[\frac{1}{2}(-2k+q)+1-j]} s^{\frac{1}{2}(-2k+q)-j} \;/.\; s \to \sigma^2\Big], k \in \texttt{Integers}\Big]$$

$$\frac{2^{\frac{1}{2}(-2k+q)}\, u^{2k}\, \sigma^{-2(j+k)+q}\, \texttt{Gamma}[1+\frac{q}{2}]\, \texttt{Gamma}[\frac{m+q}{2}]}{k!\, \texttt{Gamma}[k+\frac{m}{2}]\, \texttt{Gamma}[\frac{1}{2}(-2(-1+j+k)+q)]}$$

$$\texttt{D}[\texttt{Euq}[\sigma, u][m, q], \{\sigma^2, j\}] = \sum_{k=0}^{\infty} \texttt{foo2} =$$

$$\texttt{Simplify}\Big[\sum_{k=0}^{\infty} \frac{2^{\frac{1}{2}(-2k+q)}\, u^{2k}\, \sigma^{-2(j+k)+q}\, \texttt{Gamma}[1+\frac{q}{2}]\, \texttt{Gamma}[\frac{m+q}{2}]}{k!\, \texttt{Gamma}[k+\frac{m}{2}]\, \texttt{Gamma}[\frac{1}{2}(-2(-1+j+k)+q)]}\Big]$$

$$\Big(2^{q/2}\, \sigma^{-2j+q}\, \texttt{Gamma}\big[1+\frac{q}{2}\big]\, \texttt{Gamma}\big[\frac{m+q}{2}\big]\, \texttt{Hypergeometric1F1}\big[j-\frac{q}{2}, \frac{m}{2}, -\frac{u^2}{2\sigma^2}\big]\Big) \Big/ \Big(\texttt{Gamma}\big[\frac{m}{2}\big]\, \texttt{Gamma}\big[1-j+\frac{q}{2}\big]\Big)$$

We denote the above as

$$\texttt{DjEuq}[j\_][\sigma\_, u\_][m\_, q\_] := \frac{1}{\texttt{Gamma}[1-j+\frac{q}{2}]} \Big(\sigma^{-2j+q}\, \texttt{Echisq}[m, q]\, \texttt{Gamma}\big[1+\frac{q}{2}\big]\, \texttt{Hypergeometric1F1}\big[j-\frac{q}{2}, \frac{m}{2}, -\frac{u^2}{2\sigma^2}\big]\Big)$$

■ Fourier Transform

We consider the Fourier transformation of $h_L(u) = e^{-\frac{||u||^2}{2L^2}} ||u||^q$.

First note that

$$\texttt{Expand}\Big[-\frac{1}{2L^2}(u + i L^2 \omega)^2 - \frac{L^2 \omega^2}{2}\Big]$$

$$-\frac{u^2}{2L^2} - i u \omega$$

Thus

$$\int e^{-i\omega u}\, e^{-\frac{||u||^2}{2L^2}} ||u||^q \, du =$$

$$e^{-\frac{L^2\omega^2}{2}} \int e^{-\frac{1}{2L^2}||u + i L^2 \omega||^2} ||u||^q \, du = e^{-\frac{L^2\omega^2}{2}} (2\pi L^2)^{\frac{m}{2}} E_L\,(\,||U^*||^q \mid -i L^2 \omega) =$$

$$\texttt{foo1} = \texttt{Simplify}\Big[\texttt{PowerExpand}\Big[e^{-\frac{L^2\omega^2}{2}} (2\pi L^2)^{\frac{m}{2}}\, \texttt{Euq}[L, -i L^2 \omega, m, q]\Big]\Big]$$

$$e^{-\frac{1}{2}L^2\omega^2} L^m (2\pi)^{m/2}\, \texttt{Euq}[L, -i L^2 \omega, m, q]$$

$$H(\omega) = \frac{1}{(2\pi)^m} \int e^{-i\omega u}\, e^{-\frac{||u||^2}{2L^2}} ||u||^q \, du$$

```
Simplify[1/(2 π)^m foo1]
```

$e^{-\frac{1}{2} L^2 \omega^2} L^m (2\pi)^{-m/2}$ Euq[L, $-i L^2 \omega$, m, q]

$$= \frac{2^{\frac{1}{2}(-m+q)} L^{m+q} \pi^{-m/2} \text{Gamma}\left[\frac{m+q}{2}\right] \text{Hypergeometric1F1}\left[\frac{m+q}{2}, \frac{m}{2}, -\frac{L^2 \omega^2}{2}\right]}{\text{Gamma}\left[\frac{m}{2}\right]}$$

It simplifies to

```
Huq[ω_, L_][m_, q_] := Echisq[m, q] (2 π)^(-m/2) L^(m+q) Hypergeometric1F1[(m + q)/2, m/2, -(L^2 ω^2)/2]
```

■ Asymptotic Analysis of Fourier Transform

Let $x = \frac{L^2 \omega^2}{2}$. The last term of H (ω) becomes

```
foo1 := Hypergeometric1F1[(m + q)/2, m/2, -x]

Series[foo1, {x, ∞, 1}]
```

Series::esss : Essential singularity encountered in $e^{-\frac{1}{\frac{1}{x}}+O[\frac{1}{x}]^3}$. More…

Series::esss : Essential singularity encountered in $e^{-\frac{1}{x}+O[x]^3}$. More…

Series::esss : Essential singularity encountered in $e^{-\frac{1}{\frac{1}{x}}+O[\frac{1}{x}]^4}$. More…

General::stop : Further output of Series::esss will be suppressed during this calculation. More…

$$\text{Gamma}\left[\frac{m}{2}\right]\left(\left(\frac{1}{x}\right)^{\frac{m}{2}+\frac{q}{2}}\left(\frac{1}{\text{Gamma}[-\frac{q}{2}]}+\frac{2m+2q+mq+q^2}{4\,\text{Gamma}[-\frac{q}{2}]\,x}+O\left[\frac{1}{x}\right]^2\right)+ e^{-x}\left(\frac{1}{x}\right)^{-q/2}\left(\frac{i^q}{\text{Gamma}[\frac{m+q}{2}]}+\frac{i^q(2q-mq-q^2)}{4\,\text{Gamma}[\frac{m+q}{2}]\,x}+O\left[\frac{1}{x}\right]^2\right)\right)$$

For sufficiently large x, the dominant terms in foo1 are

```
foo2 = PowerExpand[Gamma[m/2] (1/x)^(m/2 + q/2) 1/Gamma[-q/2]]
```

$$\frac{x^{-\frac{m}{2}-\frac{q}{2}} \text{Gamma}[\frac{m}{2}]}{\text{Gamma}[-\frac{q}{2}]}$$

```
foo3 = PowerExpand[Gamma[m/2] e^(-x) (1/x)^(-q/2) i^q/Gamma[(m+q)/2]]
```

$$\frac{i^q e^{-x} x^{q/2} \text{Gamma}[\frac{m}{2}]}{\text{Gamma}[\frac{m+q}{2}]}$$

For even q, foo2=0, and only foo3 remains. Otherwise foo2 has a larger order than foo3.

See also Section 13.5.1 (p.508) of "Handbook of Mathematical Functions" M. Abramowitz and I.A. Stegun eds. (1970) Dover Publications.

■ Multiscale Bootstrap

■ Rejection Surface (linear theory)

r(u)+z = $\hat{z}(u, v) - v = \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \left(\frac{d^j E_\sigma(h(U^*) \mid u)}{d(\sigma^2)^j} \right)_{\sigma 0^2}$

For $h(u) = \|u\|^q$, we get

r(u)+z=$\sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!}$DjEuq[j][σ0,u][m,q]

We denote the rejection surface for multiscale bootstrap as RSmb

$$\text{RSmb}[k\_, \sigma 0\_][u\_][m\_, q\_] := \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \text{DjEuq}[j][\sigma 0, u][m, q]$$

■ Unbiased Surface (linear theory)

$$s(u) = \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \left(\frac{d^j E_\sigma (h(U^*) \mid u)}{d(\sigma^2)^j} \right)_{1+\sigma 0^2}$$

For $h(u) = \| u \|^q$, we get

s(u)=$\sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!}$DjEuq[j][$\sqrt{1+\sigma 0^2}$,u][m,q]

We denote the unbiased surface for multiscale bootstrap as USmb

$$\text{USmb}[k\_, \sigma 0\_][u\_][m\_, q\_] := \sum_{j=0}^{k-1} \frac{(-1-\sigma 0^2)^j}{j!} \text{DjEuq}[j]\left[\sqrt{1+\sigma 0^2}, u\right][m, q]$$

■ Bootstrap Iteration

■ Rejection Surface (linear theory)

r(u)+z $= \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!} (-1)^j E_{\sqrt{j}} (h(U^*) \mid u)$

For $h(u) = \|u\|^q$, we get

r(u)+z $= \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!} (-1)^j$Euq[$\sqrt{j}$,u][m,q]

The coefficient for j=0 is

$$\frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!}\ /.\ j\to 0$$

$-1+k$

We denote the rejection surface for bootstrap iteration as RSbi

$$\text{RSbi}[k\_][u\_][m\_, q\_] := (k-1)\,\text{Euq}[0, u][m, q] + \sum_{j=1}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!}\,(-1)^j\,\text{Euq}\left[\sqrt{j}, u\right][m, q]$$

■ Unbiased Surface (linear theory)

$$s(u) = \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!}\,(-1)^j\,E_{\sqrt{1+j}}\,(h(U^*)\ |\ u)$$

For $h(u) = \|u\|^q$, we get

$$s(u) = \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!}\,(-1)^j\,\text{Euq}\left[\sqrt{1+j}, u\right][m, q]$$

We denote the unbiased surface for bootstrap iteration as USbi

$$\text{USbi}[k\_][u\_][m\_, q\_] := \sum_{j=0}^{k} \frac{k!\,(k-1-2j)}{(j+1)!\,(k-j)!}\,(-1)^j\,\text{Euq}\left[\sqrt{1+j}, u\right][m, q]$$

■ Normal Test

■ Rejection Surface (linear theory)

The rejection surface should be parallel to the hypothesis surface, the unbiased surface should converge to the hypothesis surface for large u.

```
RSnt[u_][m_, q_] := Euq[0, u][m, q]
```

■ Unbiased Surface (linear theory)

```
USnt[u_][m_, q_] := Euq[1, u][m, q]
```

■ Multiple Comparisons

■ Rejection Surface (linear theory)

The rejection surface should be parallel to the hypothesis surface, the unbiased surface should converge to the hypothesis surface for u=0.

```
RSmc[u_][m_, q_] := Euq[0, u][m, q] - Euq[1, 0][m, q]
```

■ Unbiased Surface (linear theory)

```
USmc[u_][m_, q_] := Euq[1, u][m, q] - Euq[1, 0][m, q]
```

■ Plots of Filters and Surfaces

■ parameters

```
xmax = 3; wmax = 4;
```

```
nlist = {1, 2, 3, 4, 5, 6};
```

$$\texttt{slist} = \left\{\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right\};$$

```
mpar = 1; qpar = 1; spar1 = 1; kpar2 = 2;
```

■ hypothesis surface

■ nt

■ mc

■ def of plot1 = mb: change k

```
p1RSmb[k_] := Plot[RSmb[k, spar1][u][mpar, qpar], {u, 0, xmax}]
```

```
p1USmb[k_] := Plot[USmb[k, spar1][u][mpar, qpar], {u, 0, xmax}]
```

$$\texttt{p1RFmb[k\_] := Plot}\left[e^{\frac{w^2}{2}} \texttt{(1 - Jmb[k, spar1][w]), \{w, 0, wmax\}}\right]$$

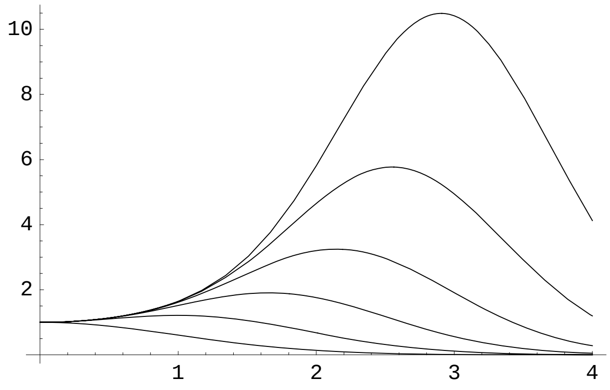
```
p1UFmb[k_] := Plot[(1 - Jmb[k, spar1][w]), {w, 0, wmax}]
```

■ run of plot1

■ plot1: multiscale bootstrap k=1,...,6 with σ0=1.

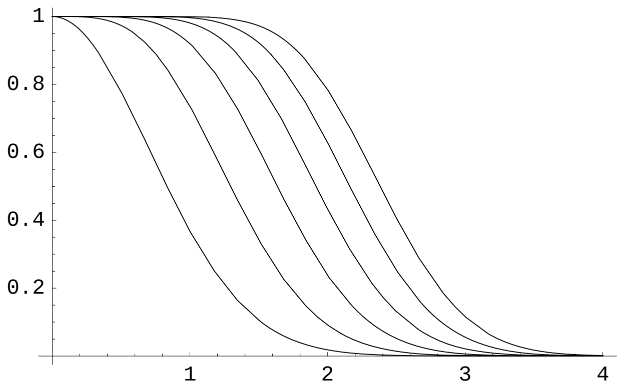
Rejection filter and Unbiased filter are

Show[**g1RFmb**]



- Graphics -

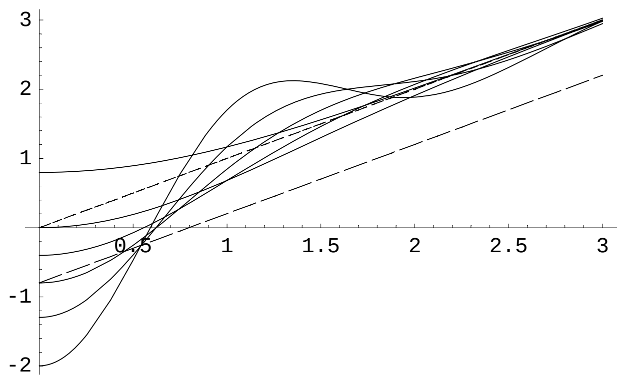
Show[**g1UFmb**]



- Graphics -

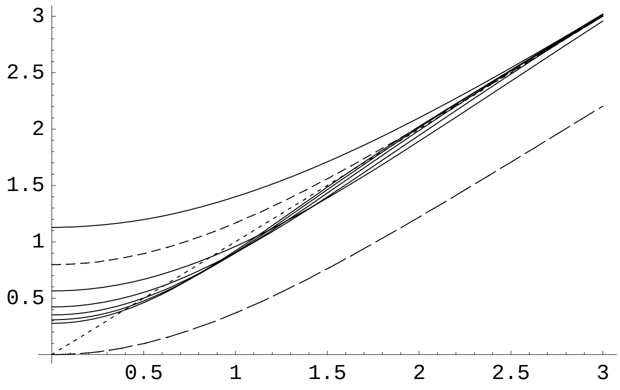
Rejection Surface and Unbiased Surface are

Show[**gh0, gRSnt, gRSmc, g1RSmb**]



- Graphics -

```
Show[gh0, gUSnt, gUSmc, g1USmb]
```



- Graphics -

■ **def of plot2 = mb: change σ**

```
p2RSmb[s_] := Plot[RSmb[kpar2, s][u][mpar, qpar], {u, 0, xmax}]

p2USmb[s_] := Plot[USmb[kpar2, s][u][mpar, qpar], {u, 0, xmax}]
```

$$\texttt{p2RFmb[s\_] := Plot}\left[e^{\frac{w^2}{2}}\ (1 - \texttt{Jmb[kpar2, s][w]}),\ \{w, 0, \texttt{wmax}\}\right]$$

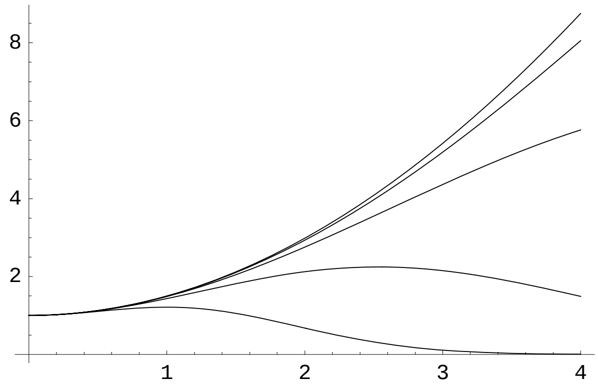
```
p2UFmb[s_] := Plot[(1 - Jmb[kpar2, s][w]), {w, 0, wmax}]
```

■ **run of plot2**

■ **plot2: multiscale bootstrap σ0=1/16,1/8,1/4,1/2,1 with k=2.**

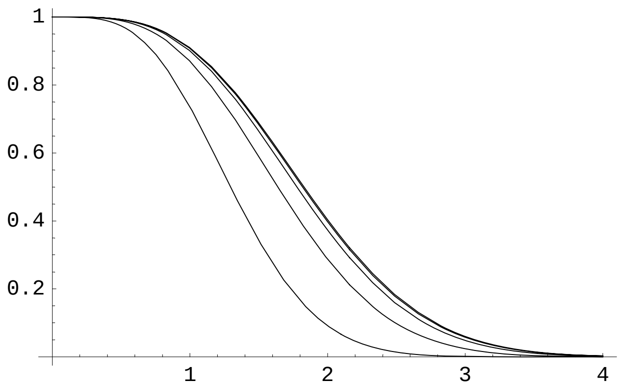
Rejection filter and Unbiased filter are

```
Show[g2RFmb, PlotRange → All]
```



- Graphics -

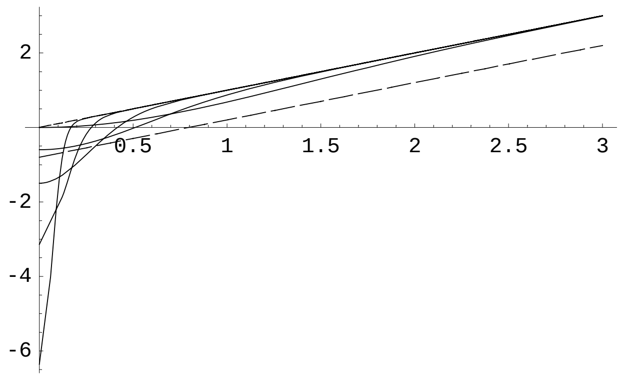
```
Show[g2UFmb]
```



- Graphics -

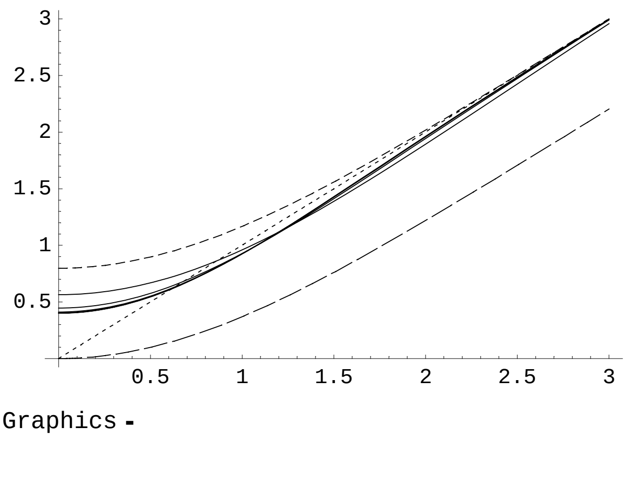
Rejection Surface and Unbiased Surface are

```
Show[gh0, gRSnt, gRSmc, g2RSmb]
```



- Graphics -

```
Show[gh0, gUSnt, gUSmc, g2USmb]
```



- Graphics -

■ def of plot3 = bi: change k

```
p3RSbi[k_] := Plot[RSbi[k][u][mpar, qpar], {u, 0, xmax}]
```

```
p3USbi[k_] := Plot[USbi[k][u][mpar, qpar], {u, 0, xmax}]
```

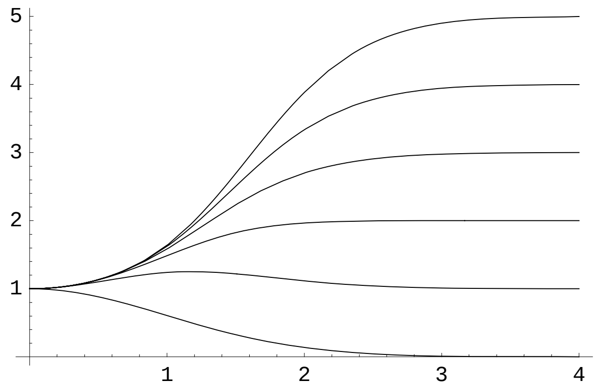
```
p3RFbi[k_] := Plot[e^(w^2/2) (1 - Jbi[k][w]), {w, 0, wmax}]

p3UFbi[k_] := Plot[(1 - Jbi[k][w]), {w, 0, wmax}]
```

- **run of plot3**

- **plot3: bootstrap iteration k=1,...,6.**

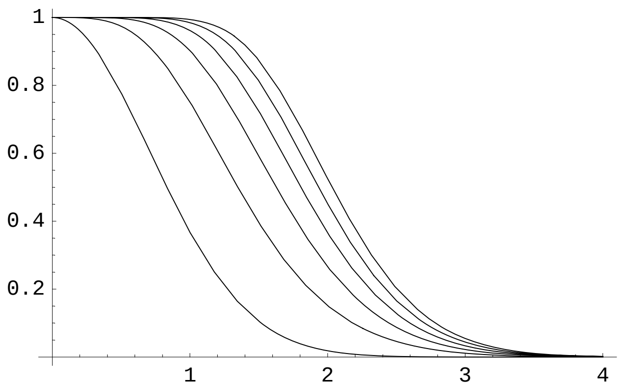
Rejection filter and Unbiased filter are

```
Show[g3RFbi]
```



- Graphics -

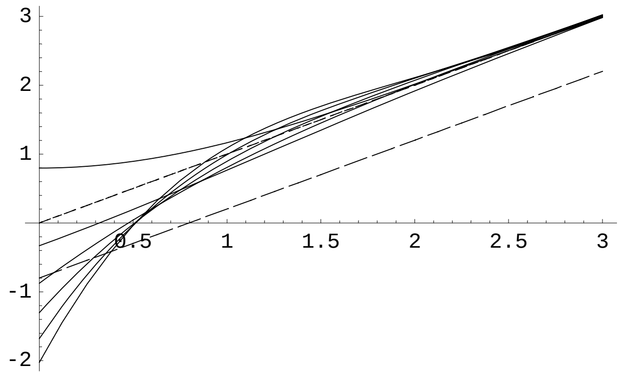
```
Show[g3UFbi]
```



- Graphics -

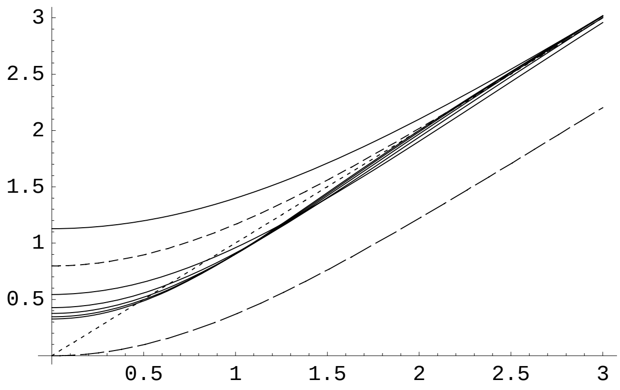
Rejection Surface and Unbiased Surface are

```
Show[gh0, gRSnt, gRSmc, g3RSbi]
```



- Graphics -

```
Show[gh0, gUSnt, gUSmc, g3USbi]
```



- Graphics -