データ解析

Rによる多変量解析入門

Rによる線形代数

lec20031023 下平英寿 shimo@is.titech.ac.jp

概要

線形代数の復習

- 行列，ベクトルの操作
- 内積と直交性
- 射影
- 行列の分解
- 一般逆行列と射影

これらの線形代数の知識を回帰分析や主成分分析で用いる

行列，ベクトルの操作

行列

$$A = \underbrace{\begin{bmatrix} a_{11} & \dots & a_{1p} \\ \cdot & & \cdot \\ \cdot & a_{ij} & \cdot \\ \cdot & & \cdot \\ a_{n1} & \dots & a_{np} \end{bmatrix}}_{p} \Bigg\} n$$

$$A = \begin{bmatrix} a^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ a^{(n)} \end{bmatrix} = [a_1, \dots, a_p]$$

$a^{(i)}$は行ベクトル，a_jは列ベクトル

セッションファイル

edu:~shimo/class/gakubu200209/note20020922.Rt

```
> A <- matrix(1:15,5)  # 5x3行列
> A
     [,1] [,2] [,3]
[1,]    1    6   11
[2,]    2    7   12
[3,]    3    8   13
[4,]    4    9   14
[5,]    5   10   15
> A[2,]  # 行ベクトルの取り出し
[1]  2  7 12
> A[,2]  # 列ベクトルの取り出し
[1]  6  7  8  9 10
> A[2,,drop=F]  # 行ベクトルの取り出し（1x3行列）
     [,1] [,2] [,3]
[1,]    2    7   12
> A[,2,drop=F]  # 列ベクトルの取り出し（5x1行列）
```

```
     [,1]
[1,]    6
[2,]    7
[3,]    8
[4,]    9
[5,]   10
> A[3:4,2:3]  # 2x2部分行列の取り出し
     [,1] [,2]
[1,]    8   13
[2,]    9   14
> B <- A  # AをBにコピー
> B[2,2] <- -1  # (2,2)要素に-1を代入
> B
     [,1] [,2] [,3]
[1,]    1    6   11
[2,]    2   -1   12
[3,]    3    8   13
[4,]    4    9   14
[5,]    5   10   15
```

```
> B[2,] <- 101:103  # 行ベクトルの代入
> B
     [,1] [,2] [,3]
[1,]    1    6   11
[2,]  101  102  103
[3,]    3    8   13
[4,]    4    9   14
[5,]    5   10   15
> B[,2] <- -1:-5  # 列ベクトルの代入
> B
     [,1] [,2] [,3]
[1,]    1   -1   11
[2,]  101   -2  103
[3,]    3   -3   13
[4,]    4   -4   14
[5,]    5   -5   15
```

```
> v <- 1:3 # ３次元ベクトル
> v
[1] 1 2 3
> as.matrix(v) # 3x1行列とみなす
     [,1]
[1,]    1
[2,]    2
[3,]    3
```

行列とベクトルの転置

$$A' = \underbrace{\begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1p} & \dots & a_{np} \end{bmatrix}}_{n} \Bigg\} p$$

$$v' = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}' = [v_1, \dots, v_n]$$

```
> t(A)
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    2    3    4    5
[2,]    6    7    8    9   10
[3,]   11   12   13   14   15
> t(as.matrix(v))
     [,1] [,2] [,3]
[1,]    1    2    3
```

行列の積

$$\underset{n\times k}{\boldsymbol{A}}\ \underset{k\times m}{\boldsymbol{B}} = \underset{n\times m}{\boldsymbol{C}}$$

$$\underset{n\times k}{\boldsymbol{A}}\ \underset{k\times 1}{\boldsymbol{v}} = \underset{n\times 1}{\boldsymbol{u}}$$

```
> A1 <- matrix(1:6,3)  # 3x2行列
> A1
     [,1] [,2]
[1,]    1    4
[2,]    2    5
[3,]    3    6
> A2 <- matrix(7:14,2)  # 2x4行列
> A2
     [,1] [,2] [,3] [,4]
```



```
[1,]    7    9   11   13
[2,]    8   10   12   14
> A1 %*% A2 -> A3  # 3x4行列
> A3
     [,1] [,2] [,3] [,4]
[1,]   39   49   59   69
[2,]   54   68   82   96
[3,]   69   87  105  123
> v1 <- 1:2  # 2次元ベクトル
> A1
     [,1] [,2]
[1,]    1    4
[2,]    2    5
[3,]    3    6
> A1 %*% as.matrix(v1)  # 3x2行列  * 2x1行列 = 3x1行列
     [,1]
[1,]    9
[2,]   12
[3,]   15
```

```
> A1 %*% v1  # ベクトルは自動的に列ベクトルとみなされる
     [,1]
[1,]    9
[2,]   12
[3,]   15
> v2 <- 1:3
> as.matrix(v2) %*% A1  # 3x1行列 * 3x2行列  はエラー
Error in as.matrix(v2) %*% A1 : non-conformable arguments
> t(as.matrix(v2)) %*% A1  # 1x3行列 * 3x2行列 = 1x2行列
     [,1] [,2]
[1,]   14   32
> v2 %*% A1  # ベクトルは自動的に行ベクトルとみなされる
     [,1] [,2]
[1,]   14   32
> v2 %*% v2 # 自動的に1x3行列 * 3x1行列とみなされる
     [,1]
[1,]   14
> v2 * v2  # 成分毎に掛け算
[1] 1 4 9
```

```
> sum(v2 * v2) # sum()は要素の和を求める関数
[1] 14
> 1:8 * 1:2  # 成分毎に掛け算，短いほうは繰り返し用いられる
[1]  1  4  3  8  5 12  7 16
> rep(1:2,4) # 1:2を4回繰り返す
[1] 1 2 1 2 1 2 1 2
> 1:8 * rep(1:2,4) # = 1:8 * 1:2
[1]  1  4  3  8  5 12  7 16
> A2
     [,1] [,2] [,3] [,4]
[1,]    7    9   11   13
[2,]    8   10   12   14
> v1
[1] 1 2
> A2 * v1 # やはり成分毎に掛け算，v1は繰り返し用いられている.
     [,1] [,2] [,3] [,4]
[1,]    7    9   11   13
[2,]   16   20   24   28
```

特殊なベクトル，行列

$$\boldsymbol{1}_n = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \Bigg\} n$$

$$\boldsymbol{I}_n = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n} \Bigg\} n$$


```
> rep(1,5)  # 成分がすべて1の5次元ベクトル
[1] 1 1 1 1 1
> as.matrix(rep(1,5))  # 5x1行列
     [,1]
[1,]    1
[2,]    1
[3,]    1
[4,]    1
[5,]    1
> diag(5)  # 5x5単位行列
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    0    0    0    0
[2,]    0    1    0    0    0
[3,]    0    0    1    0    0
[4,]    0    0    0    1    0
[5,]    0    0    0    0    1
```

対角行列と対角成分

```
> diag(1:5) # 1:5を対角成分とする5x5行列
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    0    0    0    0
[2,]    0    2    0    0    0
[3,]    0    0    3    0    0
[4,]    0    0    0    4    0
[5,]    0    0    0    0    5
> matrix(1:25,5)
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    6   11   16   21
[2,]    2    7   12   17   22
[3,]    3    8   13   18   23
[4,]    4    9   14   19   24
[5,]    5   10   15   20   25
> diag(matrix(1:25,5))  # 対角成分の取り出し
[1]  1  7 13 19 25
```


内積と直交性

ベクトルの内積

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\boldsymbol{a}'\boldsymbol{b} = \sum_{i=1}^{n} a_i b_i$$

```
> naiseki <- function(a,b) sum(a*b)
> a <- 1:5
> b <- c(1,2,1,2,1)
> a
[1] 1 2 3 4 5
> b
[1] 1 2 1 2 1
> naiseki(a,b)
[1] 21
```

ベクトルの長さ

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a}'\boldsymbol{a}} = \left(\sum_{i=1}^{n} a_i a_i \right)^{-\frac{1}{2}}$$

```
> naiseki <- function(a,b=a) sum(a*b)
> naiseki(a,b)
[1] 21
> sqrt(naiseki(a,a))
[1] 7.416198
> sqrt(naiseki(a))
[1] 7.416198
```


2 点間の距離

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$d(\boldsymbol{a},\boldsymbol{b}) = \|\boldsymbol{a} - \boldsymbol{b}\| = \left(\sum_{i=1}^{n} (a_i - b_i)^2 \right)^{-\frac{1}{2}}$$

```
> sqrt(naiseki(a-b))
[1] 4.898979
> (a-b)^2
[1]  0  0  4  4 16
> sqrt(sum((a-b)^2))
[1] 4.898979
```

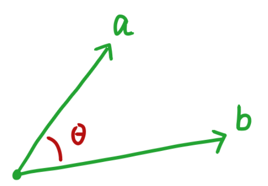

単位ベクトル

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

```
> unitvec <- function(a) a/sqrt(sum(a*a))
> unitvec(a)
[1] 0.1348400 0.2696799 0.4045199 0.5393599 0.6741999
> naiseki(unitvec(a))
[1] 1
```


ベクトル間の角度



$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\cos\theta = \left(\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \right)' \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|} = \frac{\boldsymbol{a}'\boldsymbol{b}}{\|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\|} = \frac{\sum_{i=1}^{n} a_i b_i}{\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}}$$


```
> unitvec(a)
[1] 0.1348400 0.2696799 0.4045199 0.5393599 0.6741999
> unitvec(b)
[1] 0.3015113 0.6030227 0.3015113 0.6030227 0.3015113
> naiseki(unitvec(a),unitvec(b))
[1] 0.8537714
> acos(naiseki(unitvec(a),unitvec(b)))  # ラジアン
[1] 0.5476098
> acos(naiseki(unitvec(a),unitvec(b))) * 180/pi  # 度
[1] 31.37573
```

平均，分散，共分散，相関

$a_1,\ldots,a_n$ や $b_1,\ldots,b_n$ がデータのとき

平均 $\bar{a} = \frac{1}{n}\sum a_i = \frac{1}{n}\boldsymbol{1}_n'\boldsymbol{a}$

分散 $s_a^2 = \frac{1}{n}\sum (a_i - \bar{a})^2 = \frac{1}{n}\|\boldsymbol{a} - \bar{a}\boldsymbol{1}_n\|^2$

共分散 $s_{ab} = \frac{1}{n}\sum (a_i - \bar{a})(b_i - \bar{b}) = \frac{1}{n}(\boldsymbol{a} - \bar{a}\boldsymbol{1}_n)'(\boldsymbol{b} - \bar{b}\boldsymbol{1}_n)$

相関 $r_{ab} = \frac{s_{ab}}{s_a s_b} = \frac{(\boldsymbol{a} - \bar{a}\boldsymbol{1}_n)'(\boldsymbol{b} - \bar{b}\boldsymbol{1}_n)}{\|\boldsymbol{a} - \bar{a}\boldsymbol{1}_n\|\|\boldsymbol{b} - \bar{b}\boldsymbol{1}_n\|}$

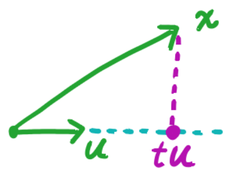
はじめに中心化 $\boldsymbol{a} \leftarrow \boldsymbol{a} - \bar{a}\boldsymbol{1}_n$ **をすると**

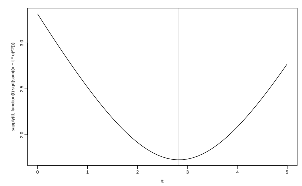
$$\bar{a} = 0, \quad s_a^2 = \frac{1}{n}\|\boldsymbol{a}\|^2, \quad s_{ab} = \frac{1}{n}\boldsymbol{a}'\boldsymbol{b}, \quad r_{ab} = \frac{\boldsymbol{a}'\boldsymbol{b}}{\|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\|}$$

c.f. 不偏分散や不偏共分散の分母は n **の代わりに** $n-1$ **を用いる**

射影

単位ベクトルのつくる直線への射影





$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

直線への射影： t **を自由に動かして** $t\boldsymbol{u}$ **が** $\boldsymbol{x}$ **へ最も近くなるようにする**

$$d(t\boldsymbol{u},\boldsymbol{x})^2 = \|t\boldsymbol{u} - \boldsymbol{x}\|^2 = \|(t\boldsymbol{u} - \boldsymbol{u}\boldsymbol{u}'\boldsymbol{x}) + (\boldsymbol{u}\boldsymbol{u}'\boldsymbol{x} - \boldsymbol{x})\|^2$$
$$= (t - \boldsymbol{u}'\boldsymbol{x})^2 + \|(\boldsymbol{I}_n - \boldsymbol{u}\boldsymbol{u}')\boldsymbol{x}\|^2$$

この最小は $t = \boldsymbol{u}'\boldsymbol{x}$ **のとき．** $\boldsymbol{x}$ **の** $\boldsymbol{u}$ **への射影は**

$$t\boldsymbol{u} = (\boldsymbol{u}'\boldsymbol{x})\boldsymbol{u} = \boldsymbol{u}\boldsymbol{u}'\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x}$$

ただし $\boldsymbol{P} = \boldsymbol{u}\boldsymbol{u}'$ **は射影行列．なお** $\cos\theta = \boldsymbol{u}'\boldsymbol{x}/\|\boldsymbol{x}\|$ **とすれば，**

$$t = \|\boldsymbol{x}\|\cos\theta$$

```
> u <- unitvec(1:5)  # 単位ベクトル
> x <- c(1,2,1,2,1)  # ベクトル
> naiseki(u,x)  # これがt
[1] 2.831639
> sum(u*x) # これでも同じ
[1] 2.831639
> tt <- seq(0,5,0.1) # 0..5
> psinit("20020922-1.eps")
> plot(tt,sapply(tt,function(t) sqrt(sum((x-t*u)^2))),type="l")
> abline(v=naiseki(u,x)) # 最小値をとるt
> dev.off()
> u*sum(u*x)  # xのu方向への射影
[1] 0.3818182 0.7636364 1.1454545 1.5272727 1.9090909
```

```
> naiseki(u,unitvec(x))  # cos(theta)
[1] 0.8537714
> acos(naiseki(u,unitvec(x))) * 180/pi  # 角度
[1] 31.37573
> P <- as.matrix(u) %*% t(as.matrix(u))  # 射影行列
> P
           [,1]       [,2]       [,3]       [,4]      [,5]
[1,] 0.01818182 0.03636364 0.05454545 0.07272727 0.0909091
[2,] 0.03636364 0.07272727 0.10909091 0.14545455 0.1818182
[3,] 0.05454545 0.10909091 0.16363636 0.21818182 0.2727273
[4,] 0.07272727 0.14545455 0.21818182 0.29090909 0.3636364
[5,] 0.09090909 0.18181818 0.27272727 0.36363636 0.4545455
> P %*% x  #  これも射影
          [,1]
[1,] 0.3818182
[2,] 0.7636364
[3,] 1.1454545
[4,] 1.5272727
[5,] 1.9090909
```

ピタゴラスの定理

$$\|t\boldsymbol{u}-\boldsymbol{x}\|^2 = (t-\boldsymbol{u}'\boldsymbol{x})^2 + \|\boldsymbol{x}-\boldsymbol{P}\boldsymbol{x}\|^2$$

まず$\boldsymbol{u}$と$\boldsymbol{x}-\boldsymbol{P}\boldsymbol{x}$は直交している．

$$\boldsymbol{u}'(\boldsymbol{x}-\boldsymbol{P}\boldsymbol{x}) = \boldsymbol{u}'(\boldsymbol{I}_n-\boldsymbol{u}\boldsymbol{u}')\boldsymbol{x} = (\boldsymbol{u}'-\boldsymbol{u}'\boldsymbol{u}\boldsymbol{u}')\boldsymbol{x} = (\boldsymbol{u}'-\boldsymbol{u}')\boldsymbol{x} = 0$$

したがって

$$\begin{aligned}\|t\boldsymbol{u}-\boldsymbol{x}\|^2 &= \|(t\boldsymbol{u}-\boldsymbol{P}\boldsymbol{x})+(\boldsymbol{P}\boldsymbol{x}-\boldsymbol{x})\|^2\\ &= \|(t-\boldsymbol{u}'\boldsymbol{x})\boldsymbol{u}+(\boldsymbol{P}\boldsymbol{x}-\boldsymbol{x})\|^2\\ &= \|(t-\boldsymbol{u}'\boldsymbol{x})\boldsymbol{u}\|^2+2(t-\boldsymbol{u}'\boldsymbol{x})\boldsymbol{u}'(\boldsymbol{x}-\boldsymbol{P}\boldsymbol{x})+\|\boldsymbol{P}\boldsymbol{x}-\boldsymbol{x}\|^2\\ &= (t-\boldsymbol{u}'\boldsymbol{x})^2\|\boldsymbol{u}\|^2+0+\|\boldsymbol{P}\boldsymbol{x}-\boldsymbol{x}\|^2\end{aligned}$$

ベクトルのつくる直線への射影

$$\boldsymbol{a} = \begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$$

直線への射影： β**を自由に動かして**$\beta\boldsymbol{a}$**が**$\boldsymbol{x}$**へ最も近くなるようにする**

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

とおけば，射影は$\boldsymbol{u}\boldsymbol{u}'\boldsymbol{x}$**である．したがって**

$$\left(\frac{\boldsymbol{a}\boldsymbol{a}'}{\|\boldsymbol{a}\|^2}\right)\boldsymbol{x} = \left(\frac{\boldsymbol{a}'\boldsymbol{x}}{\|\boldsymbol{a}\|^2}\right)\boldsymbol{a} = \beta\boldsymbol{a}$$

```
> a <- 1:5 # 射影方向
> x <- c(1,2,1,2,1)  # ベクトル
> shaei1 <- function(a,x) sum(a*x)/sum(a*a)
> shaei1(a,x)
[1] 0.3818182
> shaei1(a,x) * a
[1] 0.3818182 0.7636364 1.1454545 1.5272727 1.9090909
```


互いに直交する単位ベクトルの張る線形部分空間への射影

$$\boldsymbol{u}_1 = \begin{bmatrix}u_{11}\\ \vdots\\ u_{n1}\end{bmatrix}, \ldots, \boldsymbol{u}_p = \begin{bmatrix}u_{1p}\\ \vdots\\ u_{np}\end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$$

$$\boldsymbol{u}_i'\boldsymbol{u}_j = \delta_{ij} = \begin{cases}1 & i=j\text{のとき}\\ 0 & i\neq j\text{のとき}\end{cases}$$

$$\boldsymbol{U} = [\boldsymbol{u}_1,\ldots,\boldsymbol{u}_p]\text{は}n\times p\text{行列}, \quad \boldsymbol{U}'\boldsymbol{U} = \boldsymbol{I}_p$$

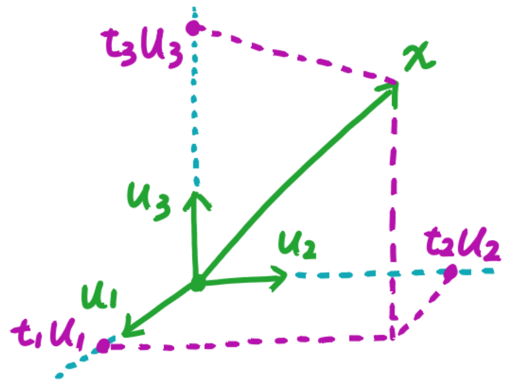
部分空間への射影： $\boldsymbol{t} = [t_1,\ldots,t_p]'$**を自由に動かして**

$$\boldsymbol{U}\boldsymbol{t} = t_1\boldsymbol{u}_1+\cdots+t_p\boldsymbol{u}_p$$

が$\boldsymbol{x}$**へ最も近くなるようにする．**

$$t_1 = \boldsymbol{u}_1'\boldsymbol{x},\ldots,t_p = \boldsymbol{u}_p'\boldsymbol{x}$$

とすればよい．



$n=3, \quad p=2$

```
> u1 <- unitvec(c(1,1,1,1)) # n=4, p=3の例
> u2 <- unitvec(c(1,1,-1,-1))
> u3 <- unitvec(c(1,-1,1,-1))
> U <- cbind(u1,u2,u3)  # 4 x 3 行列
> U
      u1   u2   u3
[1,] 0.5  0.5  0.5
[2,] 0.5  0.5 -0.5
[3,] 0.5 -0.5  0.5
[4,] 0.5 -0.5 -0.5
> t(U) %*% U  # 正規直交

    u1 u2 u3
  u1  1  0  0
  u2  0  1  0
  u3  0  0  1
> x <- c(1,2,2,2)  # ベクトル
> t(U) %*% x  # tの成分
```



```
     [,1]
  u1  3.5
  u2 -0.5
  u3 -0.5
> U %*% (t(U) %*% x)  # 射影

      [,1]
 [1,] 1.25
 [2,] 1.75
 [3,] 1.75
 [4,] 2.25
> P <- U %*% t(U) # 射影行列
> P

       [,1]  [,2]  [,3]  [,4]
 [1,]  0.75  0.25  0.25 -0.25
 [2,]  0.25  0.75 -0.25  0.25
 [3,]  0.25 -0.25  0.75  0.25
 [4,] -0.25  0.25  0.25  0.75
```

```
> px <- P %*% x  # 射影 px= U U’ x
> px

      [,1]
 [1,] 1.25
 [2,] 1.75
 [3,] 1.75
 [4,] 2.25
> qx <- x - px # 垂線
> qx

       [,1]
 [1,] -0.25
 [2,]  0.25
 [3,]  0.25
 [4,] -0.25
> naiseki(px,qx)  # pxとqxは直交
[1] 0
> Q <- diag(4) - P # 直交補空間への射影行列 Q= I - U U’
```

```
> Q

        [,1]  [,2]  [,3]  [,4]
  [1,]  0.25 -0.25 -0.25  0.25
  [2,] -0.25  0.25  0.25 -0.25
  [3,] -0.25  0.25  0.25 -0.25
  [4,]  0.25 -0.25 -0.25  0.25
> Q %*% x  # 直交補空間への射影  qx= (I - U U’)x = x - P x

        [,1]
  [1,] -0.25
  [2,]  0.25
  [3,]  0.25
  [4,] -0.25
> t(P) %*% Q  # PとQの張る空間は直交している

       [,1] [,2] [,3] [,4]
  [1,]    0    0    0    0
  [2,]    0    0    0    0
  [3,]    0    0    0    0
  [4,]    0    0    0    0
```

$$\begin{aligned}\|Ut-x\|^2 =& \|(t_1u_1-u_1u_1'x)+\cdots+(t_pu_p-u_pu_p'x)\\&+(u_1u_1'x+\cdots+u_pu_p'x-x)\|^2\\=&(t_1-u_1'x)^2+\cdots+(t_p-u_p'x)^2+\|(I_n-UU')x\|^2\end{aligned}$$

この最小は $t_1=u_1'x,\dots,t_p=u_p'x$ のとき．まとめて $t=U'x$ とも書ける．U の張る空間への x の射影は

$$Ut=UU'x=Px$$

ただし $P=UU'$ は射影行列．$Q=I_n-P$ は直交補空間への射影行列．

$$x=Px+Qx$$

と分解すると，$(Px)'(Qx)=0$ で直交している．

$$P'Q=(UU')(I_n-UU')=UU'-UU'UU'=UU'-UU'=0$$

直交変換

$$u_1=\begin{bmatrix}u_{11}\\ \vdots\\ u_{n1}\end{bmatrix},\dots,u_n=\begin{bmatrix}u_{1n}\\ \vdots\\ u_{nn}\end{bmatrix},\quad x=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix},\quad t=\begin{bmatrix}t_1\\ \vdots\\ t_n\end{bmatrix}$$

$U=[u_1,\dots,u_n]$ は $n\times n$ 行列，　$U'U=UU'=I_n$

つまり $U^{-1}=U'$ なので，

$$x=Ut=t_1u_1+\cdots+t_nu_n$$

の両辺に U' を掛けると

$$t=U'x,\quad t_i=u_i'x$$

最初の p 個と残りの $n-p$ 個に分けて考えると

$$\begin{aligned}I_n&=u_1u_1'+\cdots+u_nu_n'\\&=(u_1u_1'+\cdots+u_pu_p')+(u_{p+1}u_{p+1}'+\cdots+u_nu_n')\\&=P+Q\end{aligned}$$


```
> u1 <- unitvec(c(1,1,1,1))
> u2 <- unitvec(c(1,1,-1,-1))
> u3 <- unitvec(c(1,-1,1,-1))
> u4 <- unitvec(c(1,-1,-1,1))
> U <- cbind(u1,u2,u3,u4)
> U
      u1   u2   u3   u4
[1,] 0.5  0.5  0.5  0.5
[2,] 0.5  0.5 -0.5 -0.5
[3,] 0.5 -0.5  0.5 -0.5
[4,] 0.5 -0.5 -0.5  0.5
> t(U) %*% U

   u1 u2 u3 u4
u1  1  0  0  0
u2  0  1  0  0
u3  0  0  1  0
u4  0  0  0  1
```



```
> U %*% t(U)

     [,1] [,2] [,3] [,4]
[1,]    1    0    0    0
[2,]    0    1    0    0
[3,]    0    0    1    0
[4,]    0    0    0    1
> x <- c(1,2,2,2)  # ベクトル
> t(U) %*% x

   [,1]
u1  3.5
u2 -0.5
u3 -0.5
u4 -0.5
> U %*% (t(U) %*% x)

     [,1]
[1,]    1
```

```
[2,]    2
[3,]    2
[4,]    2
> U[,1:3]
      u1   u2   u3
[1,] 0.5  0.5  0.5
[2,] 0.5  0.5 -0.5
[3,] 0.5 -0.5  0.5
[4,] 0.5 -0.5 -0.5
> U[,4,drop=F]
       u4
[1,]  0.5
[2,] -0.5
[3,] -0.5
[4,]  0.5
> P <- U[,1:3] %*% t(U[,1:3])
> P

      [,1]  [,2]  [,3]  [,4]
```

```
[1,]  0.75  0.25  0.25 -0.25
[2,]  0.25  0.75 -0.25  0.25
[3,]  0.25 -0.25  0.75  0.25
[4,] -0.25  0.25  0.25  0.75
> Q <- U[,4,drop=F] %*% t(U[,4,drop=F])
> Q

      [,1]  [,2]  [,3]  [,4]
[1,]  0.25 -0.25 -0.25  0.25
[2,] -0.25  0.25  0.25 -0.25
[3,] -0.25  0.25  0.25 -0.25
[4,]  0.25 -0.25 -0.25  0.25
> P + Q

     [,1] [,2] [,3] [,4]
[1,]    1    0    0    0
[2,]    0    1    0    0
[3,]    0    0    1    0
[4,]    0    0    0    1
```

一次独立なベクトルの張る空間への射影

$$a_1=\begin{bmatrix}a_{11}\\ \vdots\\ a_{n1}\end{bmatrix},\dots,a_p=\begin{bmatrix}a_{1p}\\ \vdots\\ a_{np}\end{bmatrix},\quad x=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix},\quad \beta=\begin{bmatrix}\beta_1\\ \vdots\\ \beta_p\end{bmatrix}$$

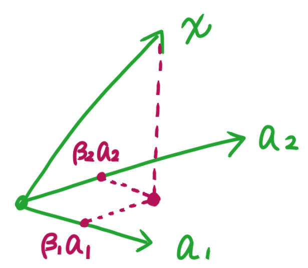
$A=[a_1,\dots,a_p]$ は $n\times p$ 行列，　$P=A(A'A)^{-1}A'$，　$PA=A$

$$\begin{aligned}\|A\beta-x\|^2&=\|(A\beta-Px)+(Px-x)\|^2\\&=\|A\beta-Px\|^2+\|Px-x\|^2\\&=\|A(\beta-(A'A)^{-1}A'x)\|^2+x'(I_n-P)x\end{aligned}$$

を最小にするには

$$\beta=(A'A)^{-1}A'x,\quad A\beta=Px$$

$P^2=P$ なので $P(I_n-P)=0$ にも注意．つまり $(Px)'(x-Px)=0$




```
> a1 <- c(1,1,1,1)
> a2 <- c(1,2,3,4)
> A <- cbind(a1,a2)
> A
     a1 a2
[1,]  1  1
[2,]  1  2
[3,]  1  3
```

```
[4,]  1  4
> B <- solve(t(A) %*% A) %*% t(A)  # solve() は逆行列
> B

   [,1] [,2]         [,3] [,4]
a1  1.0  0.5 3.885781e-16 -0.5
a2 -0.3 -0.1 1.000000e-01  0.3
> x <- c(1,2,2,2)  # ベクトル
> B %*% x  # beta

   [,1]
a1  1.0
a2  0.3
> A %*% (B %*% x)  # 射影

     [,1]
[1,]  1.3
[2,]  1.6
[3,]  1.9
```

```
  [4,]  2.2
> P <- A %*% B  # 射影行列
> P

       [,1] [,2] [,3] [,4]
  [1,]  0.7  0.4  0.1 -0.2
  [2,]  0.4  0.3  0.2  0.1
  [3,]  0.1  0.2  0.3  0.4
  [4,] -0.2  0.1  0.4  0.7
> P %*% A  # = Aのハズ

       a1 a2
  [1,]  1  1
  [2,]  1  2
  [3,]  1  3
  [4,]  1  4
> P %*% x  #  射影

       [,1]
```

```
  [1,]  1.3
  [2,]  1.6
  [3,]  1.9
  [4,]  2.2
> P %*% P

       [,1] [,2] [,3] [,4]
  [1,]  0.7  0.4  0.1 -0.2
  [2,]  0.4  0.3  0.2  0.1
  [3,]  0.1  0.2  0.3  0.4
  [4,] -0.2  0.1  0.4  0.7
> Q <- diag(4) - P
> t(P) %*% Q

                [,1]         [,2]          [,3]          [,4]
  [1,]  1.409988e-15 5.662026e-16 -2.053953e-16 -9.769949e-16
  [2,]  8.437651e-16 3.996818e-16 -1.109613e-17 -4.218868e-16
  [3,]  2.942050e-16 2.109485e-16  1.776465e-16  1.443276e-16
  [4,] -2.553432e-16 2.219904e-17  3.663790e-16  7.105454e-16
```

```
> round(t(P) %*% Q,6)

       [,1] [,2] [,3] [,4]
  [1,]    0    0    0    0
  [2,]    0    0    0    0
  [3,]    0    0    0    0
  [4,]    0    0    0    0
> t(P %*% x) %*% (x - P %*% x)

              [,1]
  [1,] 8.704138e-15
> shaei2 <- function(A,x) solve(t(A) %*% A) %*% t(A) %*% x
> shaei2(A,x)

     [,1]
  a1  1.0
  a2  0.3
> A %*% shaei2(A,x)
```

```
       [,1]
  [1,]  1.3
  [2,]  1.6
  [3,]  1.9
  [4,]  2.2
> a <- as.matrix(1:5) # 射影方向
> x <- c(1,2,1,2,1)  # ベクトル
> shaei2(a,x)  # 以前にやったshaei1(a,x)と比べよ
          [,1]
[1,] 0.3818182
> a %*% shaei2(a,x)
          [,1]
[1,] 0.3818182
[2,] 0.7636364
[3,] 1.1454545
[4,] 1.5272727
[5,] 1.9090909
```

行列の分解

特異値分解（Singular Value Decomposition）

$$n \times p \text{行列} \quad \boldsymbol{A} = \underbrace{\begin{bmatrix} a_{11} & \dots & a_{1p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \dots & a_{np} \end{bmatrix}}_{p} \Bigg\} n, \quad n \geq p$$

$$\underset{n \times p}{\boldsymbol{U} = [\boldsymbol{u}_1, \dots, \boldsymbol{u}_p]}, \quad \underset{p \times p}{\boldsymbol{V} = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_p]}$$

$$\boldsymbol{U}'\boldsymbol{U} = \boldsymbol{V}'\boldsymbol{V} = \boldsymbol{I}_p$$

$$\boldsymbol{A} = d_1 \boldsymbol{u}_1 \boldsymbol{v}_1' + \cdots + d_p \boldsymbol{u}_p \boldsymbol{v}_p'$$

$$d_1 \geq \cdots \geq d_p \geq 0$$

特異値$d_1, \dots, d_p$を対角成分にもつ行列Dを使うと

$$\boldsymbol{D} = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_p \end{bmatrix}, \quad \boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}'$$

特異値のうち0でないものの個数rが行列Aのランク（階数）である．

$$d_1 \geq \cdots \geq d_r > d_{r+1} = \cdots = d_p = 0$$

$$\boldsymbol{A} = d_1 \boldsymbol{u}_1 \boldsymbol{v}_1' + \cdots + d_r \boldsymbol{u}_r \boldsymbol{v}_r' = [\boldsymbol{u}_1, \dots, \boldsymbol{u}_r]\text{diag}(d_1, \dots, d_r)[\boldsymbol{v}_1, \dots, \boldsymbol{v}_r]'$$

```
> A <- matrix(1:15,5)  # 5x3行列
> A
     [,1] [,2] [,3]
[1,]    1    6   11
[2,]    2    7   12
[3,]    3    8   13
[4,]    4    9   14
[5,]    5   10   15
> s <- svd(A)  # 特異値分解
> s
$d
[1] 35.127223  2.465397  0.000000

$u
           [,1]        [,2]       [,3]
[1,] -0.3545571 -0.68868664  0.3656475
[2,] -0.3986964 -0.37555453 -0.3101123
[3,] -0.4428357 -0.06242242  0.0736526
```



```
[4,] -0.4869750  0.25070970 -0.6795585
[5,] -0.5311143  0.56384181  0.5503706

$v
           [,1]       [,2]       [,3]
[1,] -0.2016649  0.8903171  0.4082483
[2,] -0.5168305  0.2573316 -0.8164966
[3,] -0.8319961 -0.3756539  0.4082483

> round(t(s$u) %*% s$u,10) # U'U = I  (Uは5 x 3行列)
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
> round(t(s$v) %*% s$v,10)  # V'V = I (Vは3 x 3行列)
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
```

```
> s$u %*% diag(s$d) %*% t(s$v) # A = U D V'
     [,1] [,2] [,3]
[1,]    1    6   11
[2,]    2    7   12
[3,]    3    8   13
[4,]    4    9   14
[5,]    5   10   15
> s$u[,1:2] %*% diag(s$d[1:2]) %*% t(s$v[,1:2])  # r=2個だけ正の特異値
     [,1] [,2] [,3]
[1,]    1    6   11
[2,]    2    7   12
[3,]    3    8   13
[4,]    4    9   14
[5,]    5   10   15
```

スペクトル分解

$$G = UDU' \\ = d_1 \boldsymbol{u}_1 \boldsymbol{u}_1' + \cdots + d_n \boldsymbol{u}_n \boldsymbol{u}_n'$$

G，U，Dはすべて$n \times n$行列．
Gは対称行列 $G' = G$
Uは直交行列 $U'U = I_n$
Dは対角行列 $D = \mathrm{diag}(d_1, \ldots, d_n)$

$$G\boldsymbol{u}_i = d_i \boldsymbol{u}_i$$

なので$\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$は$G$の固有ベクトル，$d_1, \ldots, d_n$は固有値．


```
> A <- matrix(rnorm(16),4)  # 乱数をつかって行列を生成
> G <- A + t(A)  # 対称行列
> G
           [,1]       [,2]       [,3]       [,4]
[1,]  1.0199805 -1.6726827  1.7611183 -0.8899458
[2,] -1.6726827  2.7472534  0.8150037 -1.5039038
[3,]  1.7611183  0.8150037 -1.4539209 -0.5922849
[4,] -0.8899458 -1.5039038 -0.5922849 -2.1581990
> e <- eigen(G,sym=T)  # 対称行列の固有値，固有ベクトル
> e  # e$values = (d1,...,d4). e$vectors = U
$values
[1]  3.890266  1.946350 -2.498974 -3.182528

$vectors
             [,1]      [,2]        [,3]       [,4]
[1,]  0.464939132 0.7167020  0.23912371  0.4615081
[2,] -0.872833764 0.3248162  0.02187801  0.3635615
[3,]  0.003675264 0.5096684 -0.76134993 -0.4007130
```



```
[4,]  0.148254229 -0.3479523 -0.60223541  0.7030380

> t(e$vec) %*% e$vec  # U' U = I で 直交行列になっている
              [,1]          [,2]          [,3]          [,4]
[1,]  1.000000e+00  2.822822e-16 -1.313172e-16  1.542074e-16
[2,]  2.822822e-16  1.000000e+00 -1.367450e-16  2.432679e-17
[3,] -1.313172e-16 -1.367450e-16  1.000000e+00 -7.887571e-18
[4,]  1.542074e-16  2.432679e-17 -7.887571e-18  1.000000e+00
> e$vec %*% diag(e$val) %*% t(e$vec) # G = U D U' スペクトル分解
           [,1]       [,2]       [,3]       [,4]
[1,]  1.0199805 -1.6726827  1.7611183 -0.8899458
[2,] -1.6726827  2.7472534  0.8150037 -1.5039038
[3,]  1.7611183  0.8150037 -1.4539209 -0.5922849
[4,] -0.8899458 -1.5039038 -0.5922849 -2.1581990
> naiseki(e$vec %*% diag(e$val) %*% t(e$vec) - G) # つまり U D U' - G = 0
[1] 9.01027e-30
> s <- svd(G)  # 特異値分解でも同じ　（ただしベクトルの符号に注意）
> s
$d
```

```
[1] 3.890266 3.182528 2.498974 1.946350

$u
             [,1]       [,2]        [,3]       [,4]
[1,]  0.464939132 -0.4615081 -0.23912371  0.7167020
[2,] -0.872833764 -0.3635615 -0.02187801  0.3248162
[3,]  0.003675264  0.4007130  0.76134993  0.5096684
[4,]  0.148254229 -0.7030380  0.60223541 -0.3479523

$v
             [,1]       [,2]        [,3]       [,4]
[1,]  0.464939132  0.4615081  0.23912371  0.7167020
[2,] -0.872833764  0.3635615  0.02187801  0.3248162
[3,]  0.003675264 -0.4007130 -0.76134993  0.5096684
[4,]  0.148254229  0.7030380 -0.60223541 -0.3479523

> naiseki(s$u[,1] - s$v[,1])  # u1 =  v1
[1] 6.480773e-32
> naiseki(s$u[,2] + s$v[,2])  # u2 = -v2
```

```
[1] 6.594384e-31
> naiseki(s$u[,3] + s$v[,3])  # u3 = -v3
[1] 4.632945e-31
> naiseki(s$u[,4] - s$v[,4])  # u4 =  v4
[1] 6.779273e-31
> e$vec # 特異値分解のU,Vと比べよ
             [,1]       [,2]        [,3]       [,4]
[1,]  0.464939132  0.7167020  0.23912371  0.4615081
[2,] -0.872833764  0.3248162  0.02187801  0.3635615
[3,]  0.003675264  0.5096684 -0.76134993 -0.4007130
[4,]  0.148254229 -0.3479523 -0.60223541  0.7030380
> naiseki(s$u[,1] - e$vec[,1]) # u1 =  vec1
[1] 5.502171e-31
> naiseki(s$u[,2] + e$vec[,4]) # u2 = -vec4
[1] 2.249486e-31
> naiseki(s$u[,3] + e$vec[,3]) # u3 = -vec3
[1] 2.690524e-31
> naiseki(s$u[,4] - e$vec[,2]) # u4 =  vec2
[1] 5.793197e-31
```

平方根分解

対称行列Gのスペクトル分解

$$G = UDU' \\ = d_1 \boldsymbol{u}_1 \boldsymbol{u}_1' + \cdots + d_n \boldsymbol{u}_n \boldsymbol{u}_n'$$

の固有値$d_1, \ldots, d_n$がすべて非負のとき，$G^{1/2}$を次のように定義する．

$$G^{1/2} = U\mathrm{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})U' \\ = \sqrt{d_1}\boldsymbol{u}_1 \boldsymbol{u}_1' + \cdots + \sqrt{d_n}\boldsymbol{u}_n \boldsymbol{u}_n'$$

すると，$G^{1/2}G^{1/2} = I_n$である．

任意の直交行列Vを用いて$B = G^{1/2}V$とすると，

$$G = BB'$$

である．


```
> A <- matrix(1:8,4)
> A
     [,1] [,2]
[1,]    1    5
[2,]    2    6
[3,]    3    7
[4,]    4    8
> G <- t(A) %*% A  # 非負定行列
> G
     [,1] [,2]
[1,]   30   70
[2,]   70  174
> e <- eigen(G)
> e
$values
[1] 202.419122   1.580878

$vectors
          [,1]       [,2]
```

```
[1,] 0.3761682  0.9265514
[2,] 0.9265514 -0.3761682

> B <- e$vec %*% diag(sqrt(e$val)) %*% t(e$vec) # 平方根
> B
         [,1]     [,2]
[1,] 3.092629  4.52058
[2,] 4.520580 12.39211
> B %*% t(B)  # 平方根分解
     [,1] [,2]
[1,]   30   70
[2,]   70  174
```

QR分解

$$\underset{n\times p}{A} = \underset{n\times p}{Q}\ \underset{p\times p}{R}$$

$$Q'Q = I_p, \quad R\text{は上三角行列}$$

```
> A <- matrix(1:8,4)
> A
     [,1] [,2]
[1,]    1    5
[2,]    2    6
[3,]    3    7
[4,]    4    8
> q <- qr(A)
> Q <- qr.Q(q)
```



```
> Q
           [,1]          [,2]
[1,] -0.1825742 -8.164966e-01
[2,] -0.3651484 -4.082483e-01
[3,] -0.5477226 -6.163689e-17
[4,] -0.7302967  4.082483e-01
> t(Q) %*% Q
              [,1]          [,2]
[1,]  1.000000e-00 -6.776264e-18
[2,] -6.776264e-18  1.000000e-00
> R <- qr.R(q)
> R
          [,1]       [,2]
[1,] -5.477226 -12.780193
[2,]  0.000000  -3.265986
> Q %*% R
     [,1] [,2]
[1,]    1    5
[2,]    2    6
[3,]    3    7
[4,]    4    8
```

直交化とQR分解

$$A = [a_1, \dots, a_p], \quad Q = [q_1, \dots, q_p]$$

グラムシュミットの直交化: $i = 1, \dots, p$について次のステップを実行

1. $r_{ji} = q_j' a_i; \quad j = 1, \dots, i-1$
2. $b_i = a_i - \sum_{j=1}^{i-1} r_{ji} q_j$
3. $r_{ii} = \|b_i\|$
4. $q_i = b_i / r_{ii}$
5. $r_{ji} = 0; \quad j = i+1, \dots, n$

すると

$$a_i = \sum_{j=1}^{i} q_j r_{ji}, \quad A = QR$$

実際にはハウスホルダー法で求めることが多い

コレスキー分解

$$G = R'R$$

Gは$n \times n$対称行列で固有値がすべて非負，Rは上三角行列．

```
> G <- t(A) %*% A
> G
     [,1] [,2]
[1,]   30   70
[2,]   70  174
> R <- chol(G)
> R
         [,1]      [,2]
[1,] 5.477226 12.780193
[2,] 0.000000  3.265986
> t(R) %*% R
     [,1] [,2]
[1,]   30   70
[2,]   70  174
```


一般逆行列と射影

一般逆行列

$$AA^+A = A$$

とくにムーア＝ペンローズ逆行列は特異値分解を用いて次のように書ける

$$A = UDV', \quad A^+ = VD^+U'$$

$$d_1 \geq \cdots \geq d_r > d_{r+1} = \cdots = d_p = 0$$

$$D = \mathrm{diag}(d_1, \dots, d_r, 0, \dots, 0)$$

$$D^+ = \mathrm{diag}(1/d_1, \dots, 1/d_r, 0, \dots, 0)$$

$$DD^+ = D^+D = \mathrm{diag}(1, \dots, 1, 0, \dots, 0)$$

$$AA^+ = UDD^+U' = u_1u_1' + \cdots + u_ru_r'$$

$$A^+A = VD^+DV' = v_1v_1' + \cdots + v_rv_r'$$


```
> A <- matrix(1:15,5)  # 5x3行列
> s <- svd(A) # 特異値分解
> s$d #  特異値
[1] 35.127223  2.465397  0.000000
> ginvd <- function(d,tol=1e-7) {  # D+を求める関数
+   a <- d>tol
+   d[a] <- 1/d[a]
+   d[!a] <- 0
+   d
+ }
> ginvd(s$d)
[1] 0.02846795 0.40561424 0.00000000
> B <- s$v %*% diag(ginvd(s$d)) %*% t(s$u)  # Aの一般逆行列
> B
            [,1]        [,2]          [,3]        [,4]        [,5]
[1,] -0.24666667 -0.13333333 -2.000000e-02  0.09333333  0.20666667
[2,] -0.06666667 -0.03333333 -1.013136e-17  0.03333333  0.06666667
[3,]  0.11333333  0.06666667  2.000000e-02 -0.02666667 -0.07333333
```

```
> ginva <- function(A) {
+    s <- svd(A)
+    s$v %*% diag(ginvd(s$d)) %*% t(s$u)
+ }
> ginva(A)
            [,1]        [,2]          [,3]        [,4]        [,5]
[1,] -0.24666667 -0.13333333 -2.000000e-02  0.09333333  0.20666667
[2,] -0.06666667 -0.03333333 -1.013136e-17  0.03333333  0.06666667
[3,]  0.11333333  0.06666667  2.000000e-02 -0.02666667 -0.07333333
> A %*% B %*% A  # Aに等しい
     [,1] [,2] [,3]
[1,]    1    6   11
[2,]    2    7   12
[3,]    3    8   13
[4,]    4    9   14
[5,]    5   10   15
> round(A %*% B,10)  # Aの張る空間への射影行列
     [,1] [,2] [,3] [,4] [,5]
[1,]  0.6  0.4  0.2  0.0 -0.2
```

```
[2,]  0.4  0.3  0.2  0.1  0.0
[3,]  0.2  0.2  0.2  0.2  0.2
[4,]  0.0  0.1  0.2  0.3  0.4
[5,] -0.2  0.0  0.2  0.4  0.6
> round(B %*% A,10)  # A'の張る空間への射影行列
          [,1]      [,2]       [,3]
[1,]  0.8333333 0.3333333 -0.1666667
[2,]  0.3333333 0.3333333  0.3333333
[3,] -0.1666667 0.3333333  0.8333333
```

一般逆行列（$r = p$の場合）

$$
\begin{aligned}
(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}' &= (\boldsymbol{V}\boldsymbol{D}^2\boldsymbol{V}')^{-1}\boldsymbol{V}\boldsymbol{D}\boldsymbol{U}' \\
&= \boldsymbol{V}\boldsymbol{D}^{-2}\boldsymbol{D}\boldsymbol{U}' = \boldsymbol{V}\boldsymbol{D}^{-1}\boldsymbol{U}' \\
&= \boldsymbol{A}^{+}
\end{aligned}
$$

$$\boldsymbol{A}\boldsymbol{A}^{+} = \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}', \quad \boldsymbol{A}^{+}\boldsymbol{A} = \boldsymbol{I}_p$$

$$\boldsymbol{\beta} = \boldsymbol{A}^{+}\boldsymbol{x}, \quad \boldsymbol{P} = \boldsymbol{A}\boldsymbol{A}^{+}$$


```
> a1 <- c(1,1,1,1)
> a2 <- c(1,2,3,4)
> A <- cbind(a1,a2)
> A
     a1 a2
[1,]  1  1
[2,]  1  2
[3,]  1  3
[4,]  1  4
> round(solve(t(A) %*% A) %*% t(A),10)

     [,1] [,2] [,3] [,4]
  a1  1.0  0.5  0.0 -0.5
  a2 -0.3 -0.1  0.1  0.3
> round(ginva(A),10)
     [,1] [,2] [,3] [,4]
[1,]  1.0  0.5  0.0 -0.5
[2,] -0.3 -0.1  0.1  0.3
```



```
> round(ginva(A) %*% A,10)

     a1 a2
  [1,]  1  0
  [2,]  0  1
> round(A %*% ginva(A),10)

       [,1] [,2] [,3] [,4]
  [1,]  0.7  0.4  0.1 -0.2
  [2,]  0.4  0.3  0.2  0.1
  [3,]  0.1  0.2  0.3  0.4
  [4,] -0.2  0.1  0.4  0.7
```

練習問題

1-1．一般逆行列A^{+}を計算する関数`geninv`を作れ．ただし，最大特異値との比が`tol`(デフォルト値10^{-7})以下の特異値はゼロとみなす．

```
geninv <- function(A,tol=1e-7) {
ここでAの一般逆行列を計算
}
```

1-2．次の二つの行列について一般逆行列を計算し，数値計算の誤差を除いて$AA^{+}A - A = 0$となることを確かめよ．

```
A1 <- matrix(1:15,5)
A2 <- matrix(rnorm(15),5)
```

1-3．上記の二つの行列について$A^{+}A$を計算し，もし単位行列でない場合はその理由を述べよ．